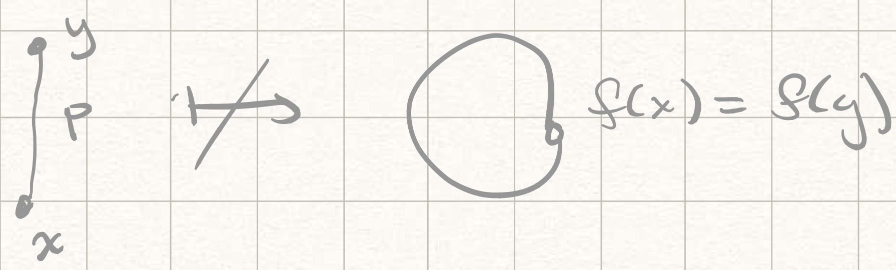
**Thm** (Gromov, Gruber)
Speaker

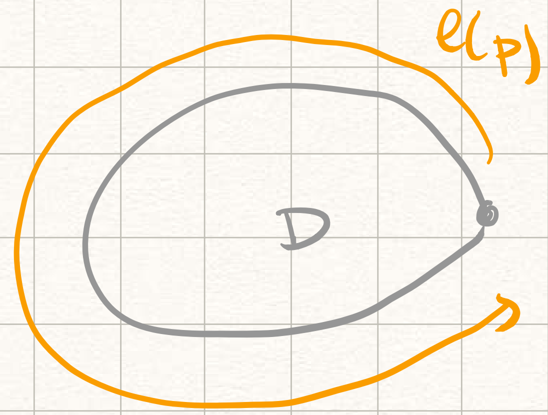
Let $\Gamma$ be a $C(6)$-labeled graph, $\Gamma_0$ a connected comp. of $\Gamma$. Then every label-preserving $f: \Gamma_0 \to Cay(G(\Gamma), S)$ is injective.

ie:

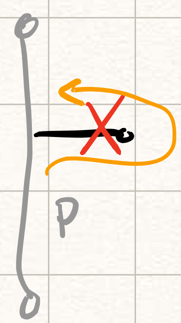


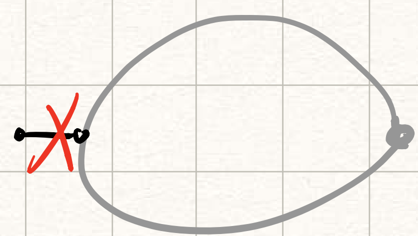

Strat: assume this happens & prove a contradiction.

Let $p$ non-closed path in $\Gamma_0$. Assume $f(p)$ closed. Let $D$ be a diagram over $\langle S \mid R_\Gamma \rangle$ w/ bound. word $\ell(p)$ st every interior arc is labeled by a $\Gamma$-piece. Among all choices for $p$, and subsequently $D$, assume that the number of edges of $D$ is minimal.

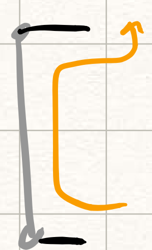


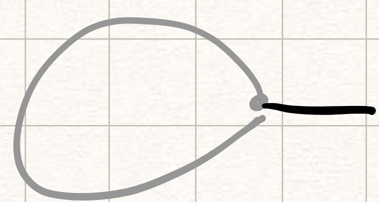

We may assume that $D$ has no spurs by minimality + reducedness of $\Gamma$.



& if at beginning can replace word.



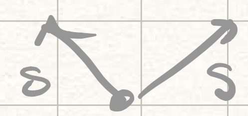


Also b/c $p$ is non-closed and $\Gamma$ reduced, $\ell(p)$ non-triv in $F(S)$.

$\Rightarrow$ $D$ has at least 1 face.

Assume $D$ has exactly one face.

Then $\ell(p) \in R_\Gamma$. Thus, we can read $\ell(p)$ on a simple closed path in $\Gamma$. In $\Gamma$, for any vertex $v$ and word $w$, there exists at most one path starting at $v$ labeled by $w$, because $\Gamma$ is reduced.




$\Rightarrow$ not red.

Therefore, there cannot exist a non-closed path and a closed path labeled by $\ell(p)$ that start at the same vertex

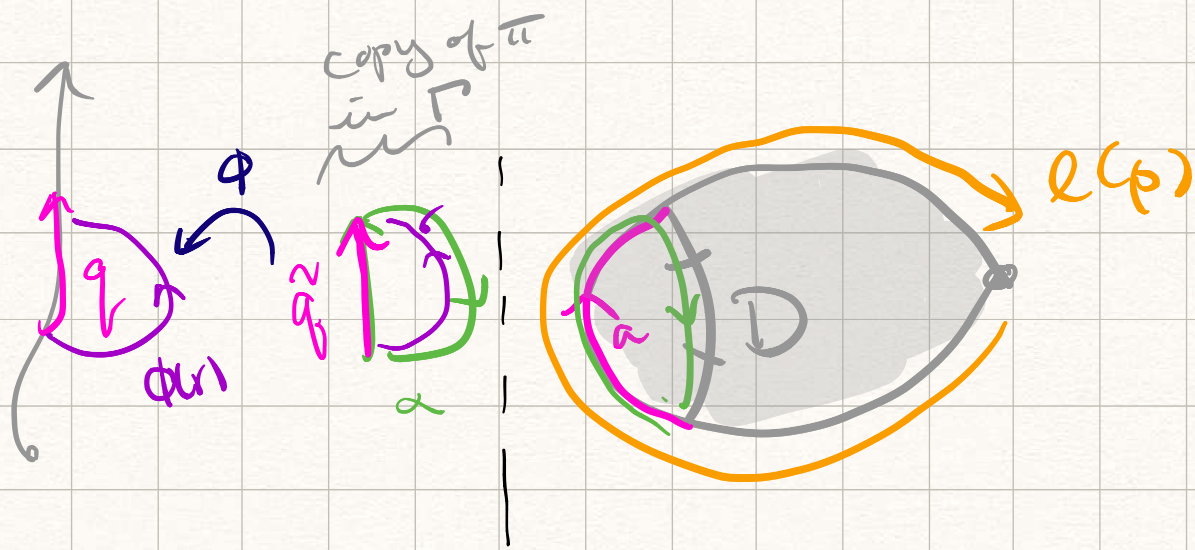
$\Rightarrow$ $\ell(p)$ is a piece.

Since $\ell(p) \in R_\Gamma$, this contradicts $C(6)$. (need at least 6 pieces)

Thus, $D$ has at least 2 faces.

$\Rightarrow$ Apply comb. Greendlinger.

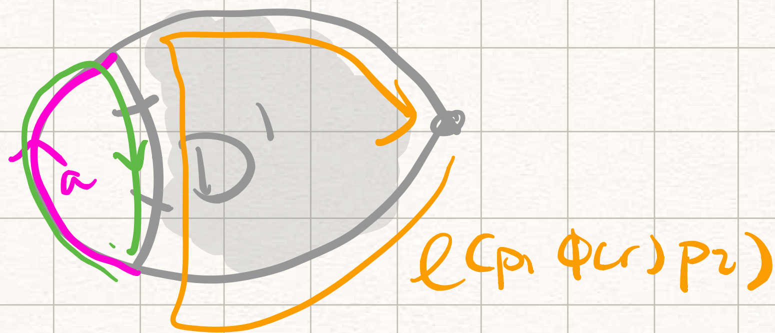
$D$ has at least one face $\pi$ w/ $e(\pi) = 1$, $i(\pi) \leq 3$ and the ext. arc $a$ of $\pi$ subpath of $\partial D$.




Assume $a$ is NOT a piece. Let $q$ be the subpath of $p$ corresponding to $a$. Let $\alpha$ be a simple closed path in $\Gamma$ w/ the same label as the $\partial\pi$. Write $\alpha = \tilde{q}r^{-1}$.
$\ell(\tilde{q}) = \ell(q) = \ell(a)$.

Since $a$ is not a piece, there exists an aut $\phi$ of $\Gamma$ st $\phi(\tilde{q}) = q$. If we write $p = p_1 q p_2$, then $p_1 \phi(r) p_2$ is a path in $\Gamma$ w/ same endpts as $p$.

Removing $a$ from $D$ gives a diag $D'$ w/ boundary word $\ell(p_1 \phi(r) p_2)$




w/ strictly fewer edges than D

Thus, $l(a)$ is a piece. Hence, $\Pi$ has a boundary word made up of $\leq 4$ $\Gamma$-pieces, contradicting $C(6)$ cond.

<u>Example of expander graph</u>

Let $(p_n)$ seq of all primes.
Let

$$T_n = \mathrm{Cay}\left(SL_2(\mathbb{Z}/p_n\mathbb{Z}),\ \left\{\begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix}\right\}\right)$$

$\Gamma = \bigsqcup_{n\in\mathbb{N}} T_n$ is an expander.

Thm (Osajdu) Let $p > 0$. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of finite connected graphs st $\exists\, C, d > 0$ w/

- $\forall n: \dfrac{\mathrm{diam}(\Gamma_n)}{\mathrm{girth}(\Gamma_n)} < C$

$\left\{\begin{array}{l} \text{diam} = \text{max path dist b/w two vertices} \\ \text{girth} = \text{shortest len of simple closed path} \end{array}\right.$

- $\forall$ vert $v$ : $d(v)$
- $\mathrm{girth}(\Gamma_{n+1}) > \mathrm{girth}(\Gamma_n)$.

Then there exist a finite set $S$ and $S$-labeling of

$$\Gamma = \bigsqcup_{n\in\mathbb{N}} \Gamma_n \text{ satisfying the}$$

$C(p)$ condition.

Rk: The expander graph $\Gamma$ in previous satisfy the condition.

**Cor**: There exists fg grp whose Cayley graph contains an expander graph $(\Gamma_n)_{n \in \mathbb{N}}$, ie. each $\Gamma_n$ is an embedded subgraph (Can use $C'(1/6)$ embedding thm due to Ollivier '06).

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## $\delta$- Hyperbolicity

(...)

**pt**: Small cancellation grps are $\delta$hyp!

**Properties of non-elementary hyp grps:**

- contain non-ab. free subgrps
- have uncountably many quotients.
- do not contain $\mathbb{Z}^2$.
- have fp.
- have soluble WP.

**Thm** (Isoperimetric ineq)

A grp $G$ is hyp $\Leftrightarrow$ it admits a finite presentation st:

- there exists $C > 0$ w/: $\forall\, w \in F(S)$ w/ $w =_G 1$, $\exists$ vK-diag $D$ w/ words $w$ st

$$|F(D)| \leq C \cdot |w| + C.$$

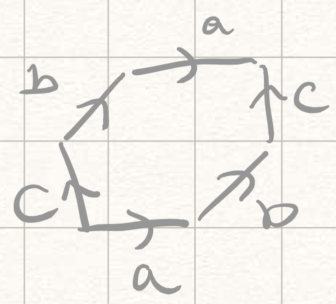
linear isoper. ineq!

**exc**: Let $\Gamma$ be a finite $C(7)$-labeled graph w/ $S$ finite. Show that $G(\Gamma)$ is hyperbolic.

**Strat**: • any $w \in F(S)$ w/ $w =_{G(\Gamma)} 1$ admits a $(3,7)$-diag over $\langle S \mid R_\Gamma \rangle$. Use curvature formula to prove linear isoperimetric ineq.

faces give strictly neg contributions.

ex: $\langle a,b,c \mid abca^{-1}b^{-1}c^{-1}\rangle$ is $C(6)$




$\cong \mathbb{Z}\oplus\mathbb{Z}^2$.

and $\langle ab, cb\rangle \cong \mathbb{Z}^2$

PT: $C(7)$ is a sharp bound on $\delta$-hyperbolicity.

Given a non-elementary hyp. grp $G$, there is a "small cancellation cond" for subsets $R \subseteq G$ st when you take the quotient $Q$ of $G$ by $\langle\langle R\rangle\rangle$, and $R$ is finite, small cancellation, then $Q$ is non-elementary hyperbolic.

**Thm** (Olshanski)

Let $G$ be a non-elementary hyp. grp, $g \in G$. Then $\exists\, k>0$ st

$G/\langle\langle g^k\rangle\rangle$ is non-elementary hyperbolic.

ex: let $w \in F(S)$, $w$ cycli. reduced, then $\langle S \mid w^6 \rangle$ is a $C'(\frac{1}{7})$ pres.

all the pieces of $G$ are proper subwords of $w$.

$\Rightarrow$ $G$ is hyperbolic.

If $|S| \geq 2$, then $G$ is non-el. hyperbolic.

**Thm** (Golod - Shafarovich '64)
There exists a fg infinite grp in which every element has finite order.

Pf: We construct grp w/ gens $\{a, b\}$,
$F(a,b) = \{g_1, g_2 \dots \}$.

By induction:
Let $G_0 = F(a,b)$, suppose we have constructed $G_n$ st $G_n$ is non-elementary hyp. and the image of each $g_i$ up to $g_n$

has finite order.

Let $G_{n+1} = G_n / \langle\langle g_{n+1}^{k_{n+1}} \rangle\rangle$,
where $k_{n+1}$ is from Olshanski (prev) theorem.

Then $G_{n+1}$ is non-elementary hyperbolic, each $g_1, \dots, g_{n+1}$ has finite order,

$$G_\infty = \langle a, b \mid g_1^{k_1}, g_2^{k_2}, \dots \rangle$$

then every element has finite ord.

$\rightsquigarrow$ Need to check that $G$ is infinite.

Suppose $G_\infty$ is finite. Then $G_\infty$ has a finite pres.

By abstract non-sense, if $G$ fp, & have $\infty$-pres, then can take finite subset

$\Rightarrow$ $G_\infty = \langle a, b \mid g_1^{k_1}, \dots g_N^{k_N} \rangle = G_N$.

$G_N$ is non-elementary $\Rightarrow$ infinite.
$\Rightarrow G_\infty$ is infinite

$\square$.

also for graphical

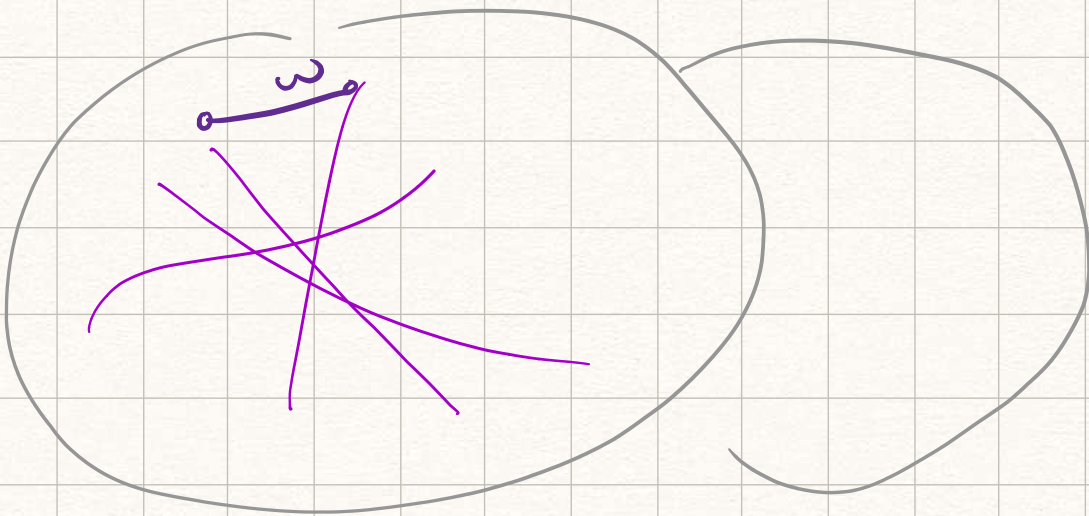
**Thm** (Speecher) Let $\langle S \mid R \rangle$ be $C(7)$ presentation w/ $|S| < \infty$, $|R| < \infty$.
Then $G$ contains a non-abelian free subgrp.

**Thm** (G + Sisto) Let $\langle S \mid R \rangle$ be as above. Then $G$ is acyl. hyp.

$\Leftrightarrow$ $G \curvearrowright$ acyl on hyp space $X$

( essentially, $G \curvearrowright$ tiling where $\delta$-$\Delta$s w/ $\delta \to \infty$.
$\rightsquigarrow$ look at dual w/ combinatorial $\Delta$s. )

If $W$ is the set of all subwords & relators, then $\mathrm{Cay}(G, S \cup W)$ is 4-hyp.




**Thm** (Coulon-G) Let $p > 0$. Then $\exists\, n_p > 0$ st $\forall\, n > n_p$ odd, if $G = \langle S \mid R \rangle$ $C'(1/6)$ pres st

- $|S| \geq 2$, no $r \in R$ is of the form $s^{\pm 1}$, $s^{\pm 1} t^{\pm 1}$ or $s^{\pm 1} t^{\mp 1}$
- no $r \in R$ is a proper power (in the $S^n$)
- no $r \in R$ has a subword of the form $u^p$,

then $G/G^n = \langle S \mid R \cup \{ w^n : w \in F(S) \} \rangle$

is infinite.

Combinatorics says: the following sequence has no subword $u^3$

ab

abba

abbabaab

abbabaab baab abba

...

$\square$

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Choose a word $r$ of len $n$ in $\{a,b\}$ unif. random.
Then as $n \to \infty$, the prob that $\langle a,b \mid r \rangle$ is $C'(1/6)$ goes to 1.

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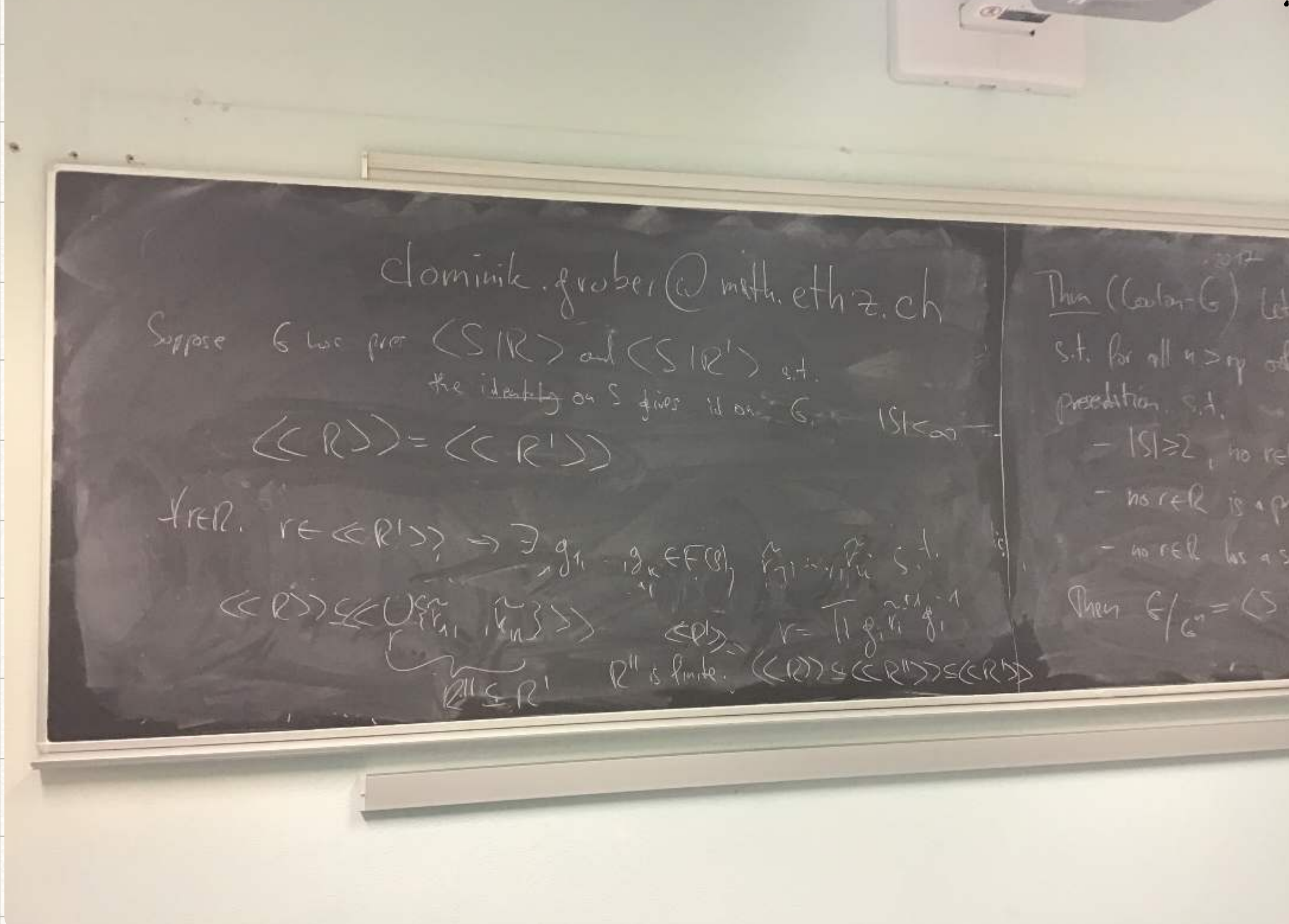
**Claim**: If $G$ is fp, & $\langle S \mid R' \rangle$ is pres st $|R'| = \infty$, then can truncate $R$.




Pf: Assume $G = \langle S \mid R \rangle$ fp.

Then $\langle\langle R \rangle\rangle = \langle\langle R' \rangle\rangle$, so

$\forall\, r \in \langle\langle R' \rangle\rangle$, $\exists\, g_1, \dots, g_n \in F(S)$, $\tilde{r}_1, \dots, \tilde{r}_n$ st

$r = \prod g_i \tilde{r}_i g_i^{-1}$

so $\langle\langle R' \rangle\rangle \subseteq \langle\langle \bigcup_r \{\tilde{r}_1, \dots, \tilde{r}_k\} \rangle\rangle$ ..

...

I'm lost !!.