WPD

<u>Def</u>: Let $G \curvearrowright S$, $h \in G$.

We say that h satisfies the <u>weakly proper discontinuous</u> (WPD) condition if $\forall \varepsilon > 0$, $\forall s \in S$, $\exists N \in \mathbb{N}$ st

$$|\{g \in G \mid d(s, gs) \leq \varepsilon \;\&\; d(h^N s, gh^N s) \leq \varepsilon\}| < \infty$$

2 <u>differences w/ acyl</u>: • <u>NOT</u> all pts, just $s, h^N s$ are compared, $\forall s \in S$.

• NOT a unif. bound like in acyl, but just that g-stabs is finite.

MAIN APPL: h is lox $\rightleftarrows$ has an axis.

"weak acyl along axis".

Thm (O + Bestvina - . .)

If G acts on S hyp & $\exists$ lox WPD element $h \in G$, then G is acyl. hyp, OR $|G : \langle h \rangle| < \infty$ (virtually cyclic).

^ much easier to check than acyl. cond.

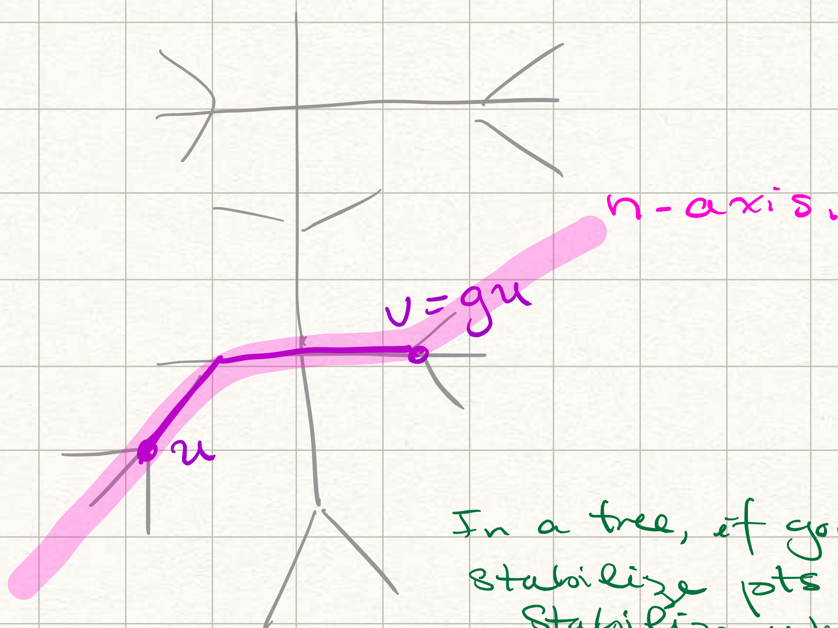
<u>ex</u>: $\mathrm{Out}(F_n)$ is acyl. hyp. for $n \geq 2$, and this comes from Thm. above.

Consider $G = A *_C B$, and suppose that C is weakly malnormal.

(**Def**: We say that $C \leq G$ is weakly malnormal if $\exists\, g \in G$ st
$$|g^{-1}Cg \cap C| < \infty.$$)

Note: every finite subgrp is weakly malnorm.

$G \curvearrowright T$ Bass-Serre tree.



In a tree, if you good.
stabilize pts on segment,
Stabilize whole segment.

Weak malnormality
$\Rightarrow |\mathrm{Stab}_G(\{u,v\})| < \infty$

Stab of first edge in G
Stab other in $g^{-1}Cg$.
$\Rightarrow$ malnormal = finite

Obs

No proper invariant subtrees.

If $G \overset{\curvearrowright}{\text{min}} T$, $\mathrm{Fix}_G(\partial T) = \emptyset$,
then $\forall u, v \in V(T)$, $\exists$ lox $h \in G$
st $\mathrm{axis}(h) \supseteq \{u, v\}$.

This implies that $h \in$ WPD lox elmt.

Thm: (M-O)

(a) Assume $G = A \ast_C B$, where $A \neq C \neq B$ and C is weakly malnormal in G.

looser than in A or B.

Then G is acyl. hyp or virtually cyclic.

(b) Let G = HNN ext
$$= A \ast_{C^t = D}$$

where C is weakly malnormal in G. Then G is acyl. hyp.

ex: $\mathrm{Aut}(K\langle x, y\rangle)$ is acyl. hyp for any field K.

free associative alg over field K

(b/c know decomp as $A \ast_C B$ where C malnormal).

ex: (HW): $H = \langle a,b,c,d \mid a^b = a^2,\ b^c = b^2,\ c^d = c^2,\ d^a = d^2 \rangle$

Higman grp

is acyl. hyp.

Thm (M-Osin): If $G = \pi_1(M)$ where M is a connected, compact, orientable, (irreducible), manifold, then exactly one of the following holds:

can always assume

(1) G is acyl. hyp

(2) M is Seifert-Fibered

(3) G is virt. solvable.

Other examples of acyl. hyp. grps

- **Thm(0)**: Grps of deficiency ≥ 2, ie, $\exists\, G = \langle X \mid R \rangle$, $|X|, |R| < \infty$ and $|X| - |R| \geq 2$. are acyl. hyp.

WAT?!? pf uses vN-algebras & Betti numbers.

idea: can use def to decompose as HNN-ext.

then: result by Thompson & al
says that for certain
Betti-number subgrp fg
& infinite index
= weakly malnormal.

Open Problem: Find elementary pf of this
Thm (ie, minimize analytic
tools)

- $C'(1/6)$ non-cyclic (Gruber & Sisto)
- Non-elementary convergence grps. (Sun)
generalizes hyp.

Properties

1. Every acyl. hyp. grp has a
finite amenable radical $K(G)$
(max normal amenable subgrp)
note: - no chain of ascending size here.
- $K(G)$ = ker of action on ∂S.
↳ well-defined here.

2. Acyl. hyp ⇒ not inner amenable.

- already NOT amenable b/c contains free subgrp. F_2
- inner-amenability weaker than am. request that measure inv. under conj.

3. If $\forall$ a.h. G is NOT boundedly gen (product of fin. many cyclic subgrps)

ex: $SL_3(\mathbb{Z})$ – gen. by elementary mat.

More generally, if $G = A_1, \dots, A_k$, then $\exists i$ st A_i is a.h.

of course, then G cannot be such a prod of cyclic grps.

Model cases

Hyp	Rel. hyp. grp	Acyl. hyp
F_n	$A * B$ A, B satisfy P ⇒ $A * B$ satisfies P	Whatever you can prove about $A*B$ w/o any info about A, B (except non-trivial.

Free prods SQ-univ.

subgrp of quotient.

Def: G is SQ-universal if $\forall$ every countable grp embeds in a quotient of G.

NON-ex = amenable, finite, any prop. stable under quotients.

ex =
- F_2 is SQ-universal
- Hyperb. non-ele ⇒ SQ-univ

"SQ = large enough in an alg. sense":

SQ $\Rightarrow$ $G \geq F_2$

$\left\{\begin{array}{l} \text{SQ} \\ +\text{fg} \end{array}\right.$ $\Rightarrow$ G has $\underbrace{2^{\aleph_0}}$ normal subgrps.

uncountably many. $\Rightarrow$ uncountably many quotients.

Thm: Every acyl. hyp. grp is SQ-univ.

"acyl grps are very large".

Thm: Let Γ be a lattice in a connected Lie-grp. If Γ is acyl hyp, then it is commensurable to a lattice in a non-cmpct simple Lie grp of rk 1. In particular, Γ is non-elementary relatively hyperbolic.

Benson-Farb thm.

so the world of lattices don't expand w/ AH.

Pf: Γ is a lattice in G.

S = solvable radical of G.

$\Rightarrow$ $\underbrace{S \cap \Gamma}_{\text{finite}}$ is a lattice in Γ.

and $\Gamma/(S\cap\Gamma)$ is a lattice in G/S - semi simple

$= M_1 \times M_2 \times \ldots \times M_k$.

WLOG Γ is a lattice in $M_1 \times \dots \times M_k$

simple

① $\exists \geq 2$ non-cmpct factors.

Mond-Shalom: $H^2_b(\Gamma, \ell^2(\Gamma)) = 0$

Hamenstadt: Γ is acyl. hyp

$\Rightarrow H^2_b(\Gamma, \ell^2(\Gamma)) = \infty$

impossible

② $\exists \leq 1$ non-cmpct factor.

Γ is commensurable to a lattice in a simple Lie grp M.

Margulis normal subgrp thm + SQ

$\Rightarrow$ $\forall$ higher rank lattice has countably many normal subgrps.

Thm (Sisto)

Let $G \curvearrowright$ acyl δ-hyp be f.g. Then, the probability that a simple random walk arrives at a lox. elmt after n-steps is:

$$1 - O(\varepsilon^n), \quad \varepsilon \in (0,1).$$

Techniques

(1) Hyperbolically embedded subgrps.
[Dah + O + ..]
(i) Grp theoretic Dehn filling.

(2) Small Cancellation Theory (Hull 2016)

(3) Monod - Shalom rigidity theory for measure pres. actions.

Acyl. hyp $\Rightarrow$ C_{reg}
"
$\left\{ \begin{array}{l} \text{Grps satisfying} \\ H_b(\Gamma, \ell^2(\Gamma)) \neq 0 \end{array} \right\}$

<u>Note</u>: There is a Survey
"Grps acting acyl. on hyp. spaces"
20 pages + refs to results & tech.

Good start ☺.

Open Questions

(1) Is Acyl. hyp. a q-isom invariant of fg grp?
(If G qi H, and G acyl. hyp, is H acyl. hyp?)

(1') Let G be ah. $G \leq H$, $[H:G] < \infty$.
Is H ah? Not known, even for index 2.

The obvious thing doesn't work for H is infinitely gens.
$\rightsquigarrow$ the most interesting cases are infinitely gens.

(2) Background
$\beta_1^{(2)}(G) > 0$

$\Leftrightarrow$ G is non-amenable &
$\exists$ unbounded $q: G \to \ell^2(G)$ st
$\forall g, h \in G$,
$q(gh) = q(g) + \lambda_G(g) q(h)$.

Q: $\beta_1^{(2)}(G) > 0$ & G fp $\overset{?}{\Longrightarrow}$ G is a.h.

Rmk: Not true for ∞-presented grps. (Chück - O)

(3) Is acyl. hyp. a measure equivalent - invariant?

(M-S) $H_b^2(G, \ell^2(G)) \neq 0$ is a measure - equivalent invariant.

we don't know a single grp which is NOT acyl & satisfies this.