Small Cancellation

What it's good for:

1) Construct fg ∞-grps which has exceptional properties.

ex:
- ∞ simple grps
- ∞ grps where every element has finite order.
- non cyclic grps where every proper subgrp is cyclic.
- Infinite grps where any two non-trivial elmts are conjugates.
- Fg grps whose Cayley graph contain expander graphs.

2) Structure Theory for negative curvature.

ex:
- Gromov hyperbolic (e.g. fundamental grps of closed manifolds for ct negative sectional curvature).

alg. props: - non-ab free subgrps,
- uncountably many quotients.

- (un)solvability & decision problems.

Plan:

1) Classical small cancellation theory
   - Construction of ∞-grps, $\underbrace{\text{combinatorial}}_{\text{finite planar graphs.}}$ npc.

2) Graphical small cancellation (s.c.)
   constructing Cayley graphs containing prescribed subgraphs.

3) Connections to hyperbolicity.

I - Classical SCT

ex 1: An infinite fg grp w/ non-trivial finite quotients.

let $G = \langle a, b \mid r_1, \rho_1, r_2, \rho_2 \ldots \rangle$
st $r_n = a_1^{-1} \underline{v_n(a^n, b^n)}$

ie $a^n (b^n)^2 (a^n)^3$

ie word w/ gens a^n, b^n.

$v \in \langle a^n, b^n \rangle \in F(a,b)$.

$$\rho_n = b^{-1} w_n(a^n, b^n)$$

for some words w_n, v_n.

Claim: G has no non-trivial finite quot.

Pf: Let $\pi : G \to F$ be a surjective hom. $|F| < \infty$.
Denote $N := |F|$.
We know $f \in F \Rightarrow f^N = 1$.
We have

$$\pi(a) = \pi(\underline{v_n(a^n, b^n)})$$

$a^{-1} v_n(a^n, b^n) = 1$

fix $n = N$.

$$= \pi(v_N(a^N, b^N)).$$
$$= 1$$

$$\pi(b) = \pi(w_n(a^n, b^n)) = \pi(w_n(a^n, b^n)) = 1$$

$\Rightarrow \pi(G) = \{1\} = F.$

Problem: Is G ∞? Don't know if we didn't just construct trivial grp.

$\Rightarrow$ Use SCT! ($\star$)

Notation: if $R \subseteq F(S)$ is a set of cyclically reduced words

w is reduced if all of its cyclic shifts are reduced.

e.g.: abc^{-1} ✓

aba^{-1} ✗ b/c $a^{-1}ab$ ✗.

Let $\bar{R}$ be the set of all cyclic shifts of elmts of $R \cup R^{-1}$.

Then: $\langle\langle R \rangle\rangle = \langle\langle \bar{R} \rangle\rangle$

because conj = shift.

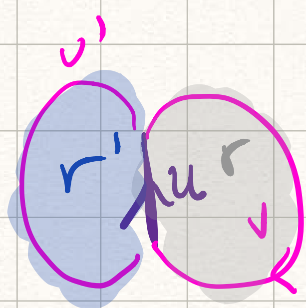
We say that R is concise if no two elmts of $R \cup R^{-1}$ are cyclic shifts of one another (ie, if you have $w \in R$, then no conj of w is in R).

Def: (Small cancellation condition)

Let $\langle S \mid R \rangle$ be a pres.

A piece is a word of $u \in F(S)$ st $\exists\, r \neq r'$ in R st

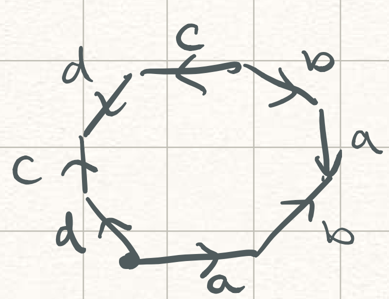
two different ways. $\begin{cases} r \equiv uv \\ r' \equiv uv' \end{cases}$ equality of words.



Let $n \in \mathbb{N}$. Then $\langle S \mid R \rangle$ satisfies the $C(n)$ condition if

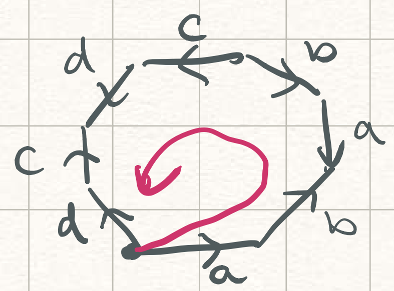
- R is concise
- every $r \in R$ is cyclically reduced
- no $r \in \bar{R}$ is a product of strictly fewer than n pieces.

ex's: 1) $\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

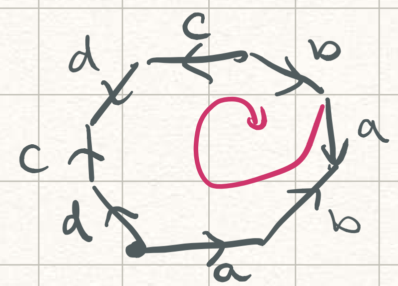


pieces starting w/ a

$\bar{R}$: • $\underline{a}ba^{-1}b^{-1}cdc^{-1}d^{-1}$,



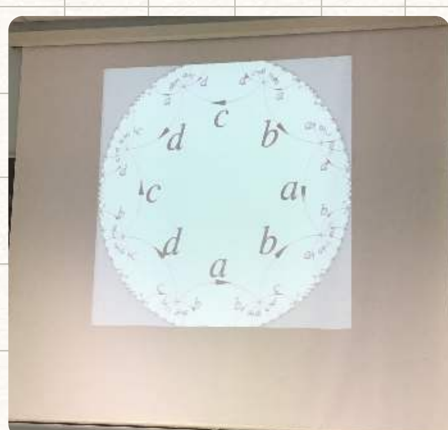
• $\underline{a}b^{-1}a^{-1}dcd^{-1}c^{-1}b$



$\Rightarrow$ a is a piece, but ab is **NOT** a piece.

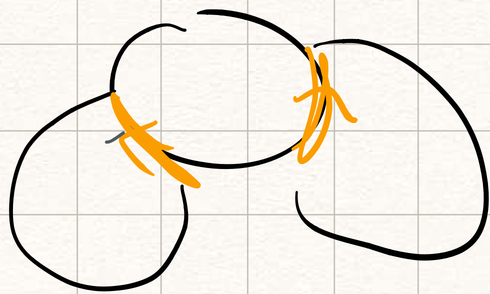
$\rightsquigarrow$ No word of len ≤ 1,

Also: $C(8)$ is satisfied.



Idea of piece

have relator & get Cay(G)



overlapping elmts are pieces.

cancellation is big if pieces are small!

2) $\langle a, b \mid a^n, b^m \rangle$

a^n = only elmt in $\bar{R}$ starting w/ a.

$\Rightarrow$ No piece starting w/ a.

b^m = " "

$\Rightarrow$ " " .

$\Rightarrow$ No non-empty piece.

$\rightsquigarrow$ this satisfies every $C(p)$ condition (pieces are infinitely small).

3) $\langle a,b \mid (a^n b^n)^n : n \geq k \rangle$ for k fixed.

exc: all pieces have the form $a^m b^\ell$ or $b^m a^\ell$ or inverses + subwords.

$\rightsquigarrow$ satisfies $C(k)$ condition.

PI: easy to write down presentations which satisfy $C(n)$.

Thm: Suppose G is given by a $C(6)$ condition $\langle S \mid R \rangle$.
Let u be a proper, non-empty subword of some $r \in \bar{R}$. Then $u \neq 1$ in G.

Cor: If $\exists\, r \in R$ st $|r| \geq 2$, then G is non-trivial. (*)

★: Back to ex 1: Suppose $\langle a,b \mid r_n, \rho_n : n \in \mathbb{N} \rangle$ is $C(6)$, and w_n, U_n non-empty.

Since G is its own non-trivial quot, G is infinite.

★: $\rightsquigarrow$ we know that G is non-trivial b/c it has relations that are longer than 2. (assuming G is $C(6)$ by choosing v_n, w_n). + G doesn't have finite non-triv quotient $\Rightarrow |G| = \infty$.

<u>Rk</u>: Can promote G to an infinite fg simple grp via:

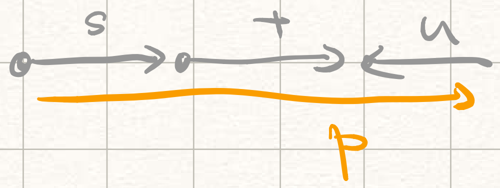
<u>Lemma</u>: Let G be a fg infinite grp. Then G contain a maximal (wrt $\subseteq$) proper normal subgrp N.
Moreover, G/N is simple.

Thus, the quotient of G by its max. proper subgrp is non-trivial + simple.

Because <u>it's</u> non-trivial, it's infinite.
G/N $\Rightarrow$ so replace G by G/N.

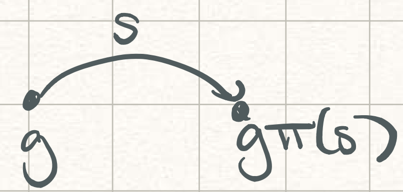
<u>Def</u>: Let S be a set. An <u>S-labeled graph</u> is a graph Γ together w/ a choice of each edge of
- an elmt of S
- an orientation.

If p is a path, then $\rho(p)$ is the word read on p.

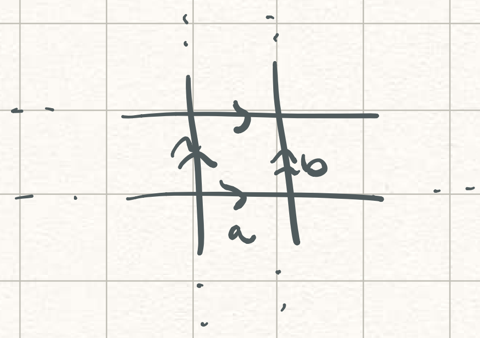


$\rho(p) = s t u^{-1}$

ex: Let $\pi : F(S) \to G$ be an epimorphism. Then $Cay(G, S)$ is the S-labelled graph w/ vertex set G, and for $g \in G$, $s \in S$ draw edge

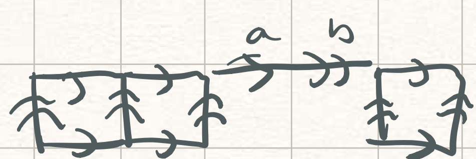


ex: $\langle a, b \mid a b a^{-1} b^{-1} \rangle$



Van Kampen diag:

glue relators together & connect.



Def: Let S set. An **S-labeled van Kampen diagram** D is a finite, connected S-labelled graph w/:

- a chosen embedding in $\mathbb{R}^2$,
- a chosen path ∂D traversing the boundary of D once called "**boundary path**"

A bounded region of D is called a **face**.

A **positive (resp negative) boundary path** of a face Π is a path traversing the boundary of Π w/ positive (resp. negative) orientation.

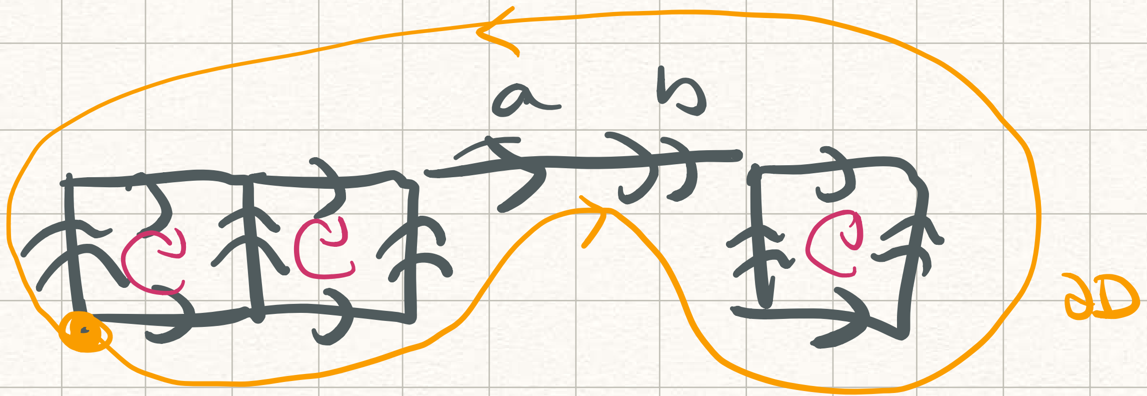
$\overset{\shortparallel}{\text{clockwise (cw)}}$ $\overset{\shortparallel}{\text{counter-cw.}}$

A boundary word of π is a word that can be read as one of the boundary paths of π.

Let $\langle S | R \rangle$ be a presentation.

A van Kampen diag. over $\langle S | R \rangle$ is an S-labelled diag where every face is a bounded word in R.

ex:

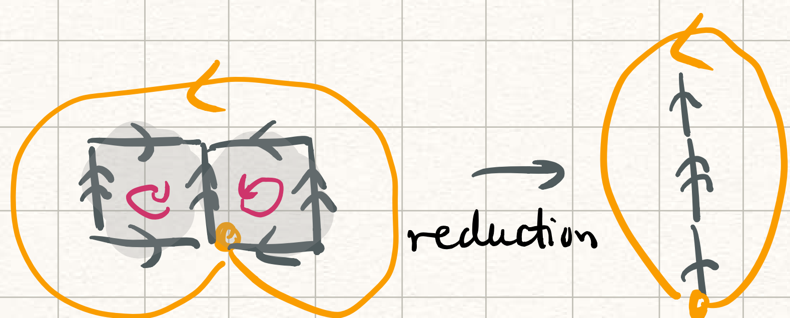


The label of ∂D is:

$w = a^2 b a b b^{-1} a b a^{-1} b^{-1} a^{-3} b^{-1}$

(Note: every word has a VK diag.)

- We say ∂D is reduced if there does NOT exist an edge e in D in the intersection of faces Π_1, Π_2 st Π_1 has a pos. bound. path p_1 starting w/ e, Π_2 has neg. bound. path starting w/ e st the labels of p_1 & p_2 are equal in $F(S)$.



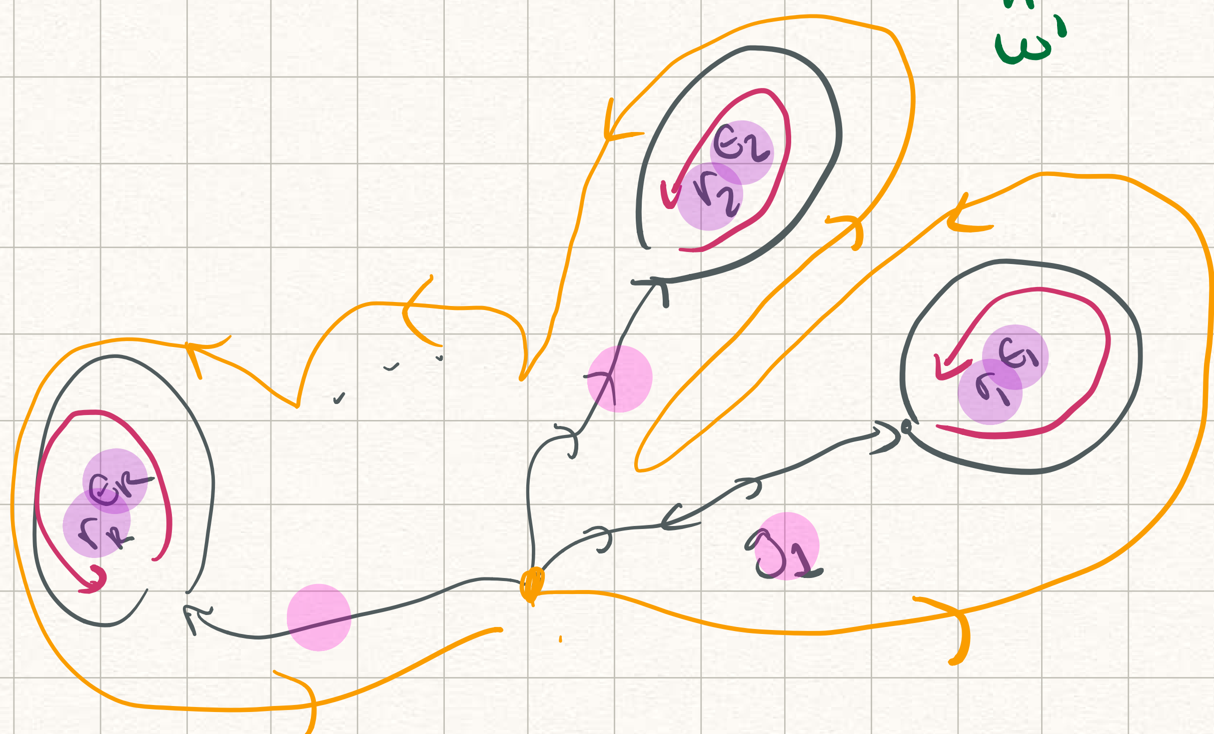
Thm: (van Kampen's lemma)

Let $\langle S \mid R \rangle$ be a pres. of G and $w \in F(S)$. Then $w = 1$ in $G \iff$ there exists a reduced van Kampen diag D over $\langle S \mid R \rangle$ w/ boundary word w.

(Sketch pf): Suppose $w = 1$ in G.

Then $w \in \langle\langle R \rangle\rangle$, ie, $\exists\ r_1, \ldots, r_k \in R$, $\epsilon_1, \ldots, \epsilon_k \in \{+1, -1\}$, $g_1, \ldots, g_k \in F(S)$ st

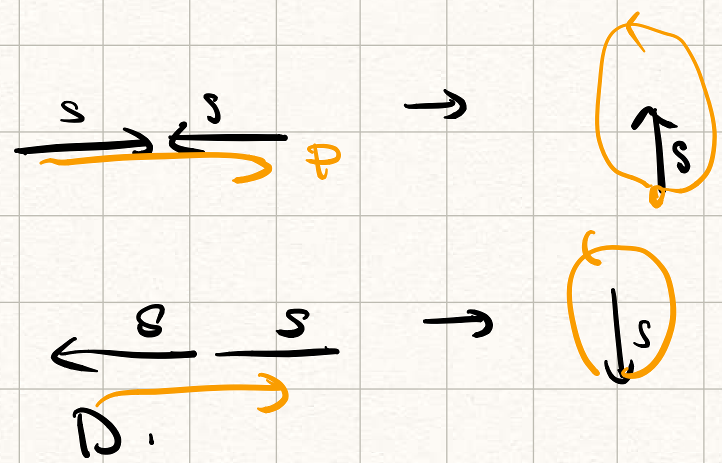
$$w =_{F(S)} \underbrace{\prod g_i r_i^{\epsilon_i} g_i^{-1}}_{= w'}$$



- If w' is reduced, then $w = w'$ b/c $w \in F(S)$ = set of red. words in S.

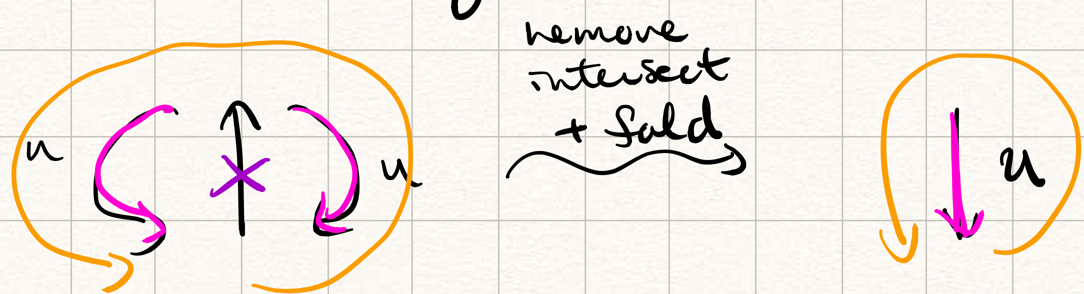
- If w not reduced, then there are two conseq. edges $\xrightarrow{s}\xleftarrow{s}$ p or $\xleftarrow{s}\ \xrightarrow{s}$ in the boundary of D.

  $\Rightarrow$ Fold away edges to shorten the rs.

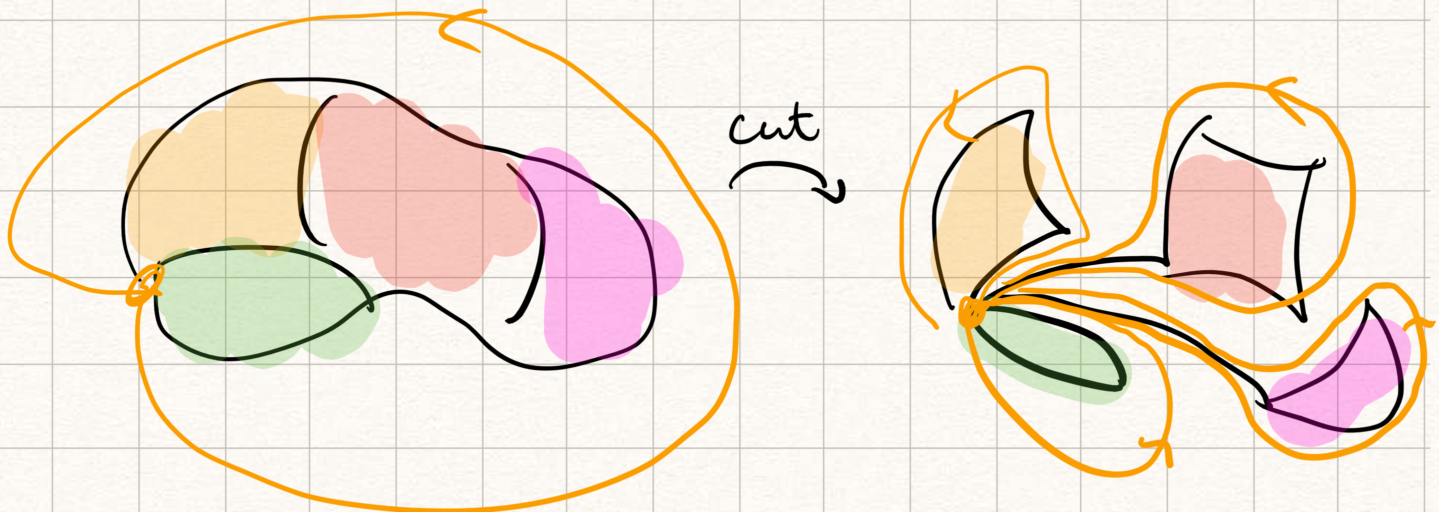


Fold away such edges to shorten the boundary. Keep going until reduced.

→ To make vk-diag reduced,



For the converse, if D is VK over $\langle S | R \rangle$ w/ bound. word w, then $w =_G 1$ b/c:



cut up into lollipop diag. w/ freely equal boundary word w'

$\Rightarrow$ w' is in $\langle\langle R \rangle\rangle$

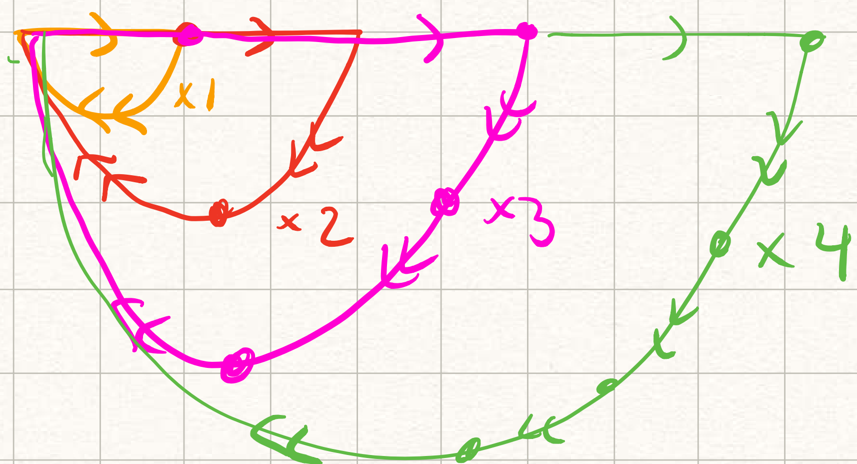
$\Rightarrow$ so is w.

Exercises

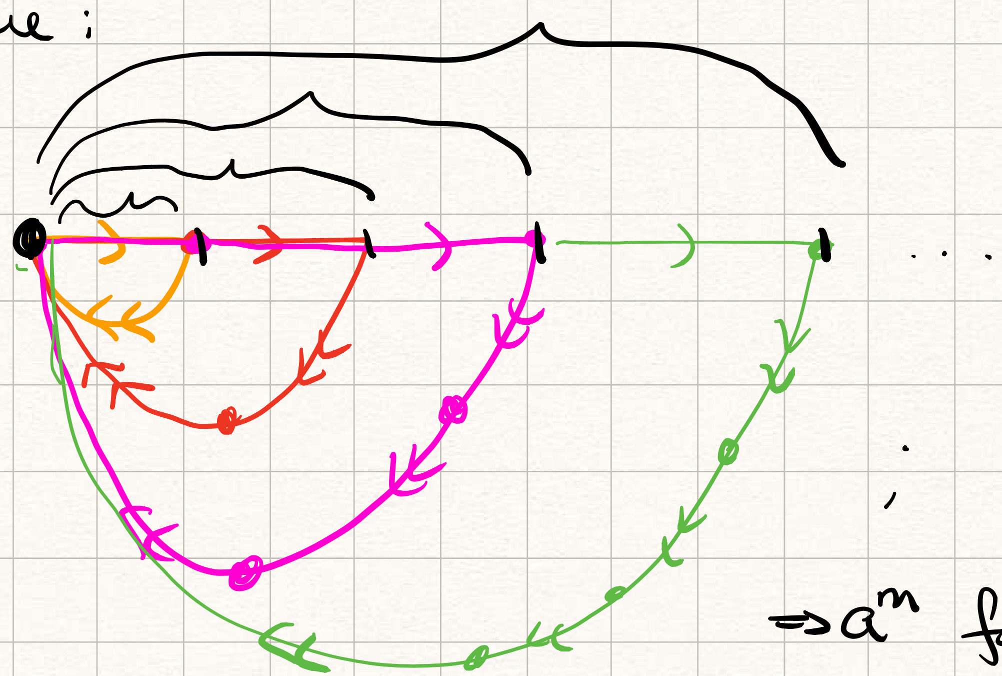
i) $\langle a, b \mid (a^n b^n)^n : n \geq k \rangle$ for k fixed.

exc: all pieces have the form $a^m b^l$ or $b^m a^l$ or inverses + subwords. satisfies $C(k)$.

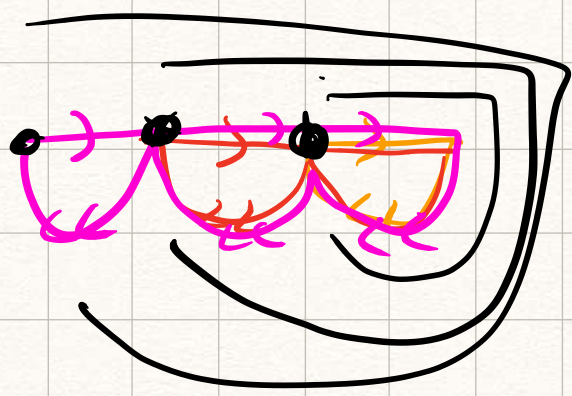
relations of the form $(a^n b^n)^n$ look like this:



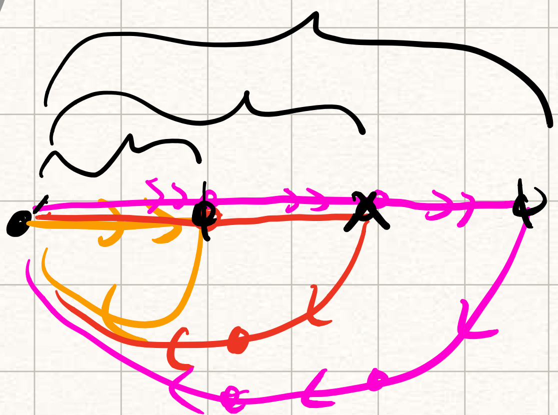
pieces are:



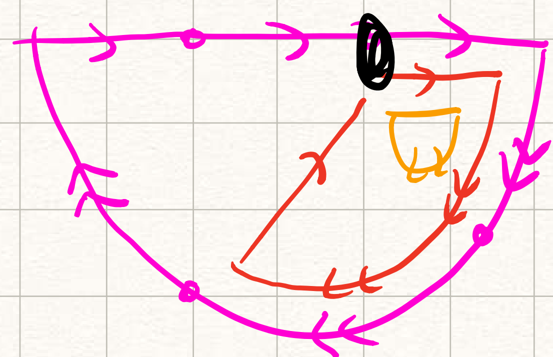
this is just 1-iteration of the $n \geq k$ power in $(a^n b^n)^n$



$\Rightarrow a^m b^m$ form



b^m form



$a^m b^l$ form.

this just depends on the cyclic permutation of the relations. (ie start pts).

why is it $C(k)$?

Let $n \in \mathbb{N}$. Then $\langle S \mid R \rangle$ satisfies the $C(n)$ condition if
- R is concise
- every $r \in R$ is cyclically reduced
- no $r \in \bar{R}$ is a product of fewer than n pieces.

strictly

if $n \geq k$, then we have k loops, and at least 1 piece in each loop

$\rightsquigarrow$ so no relation can use less than k pieces.

$\Rightarrow$ $C(k)$ satisfied. $\square$