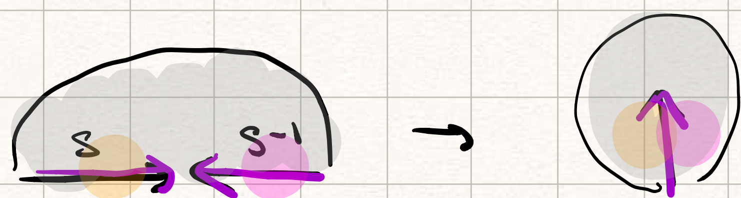
van Kampen diag

let $\langle S \mid R \rangle$ pres. Then a vK-diag D over $\langle S \mid R \rangle$ is a finite connected S-labelled graph embedded in $\mathbb{R}^2$ st every bounded region ("face") has a boundary word in R.

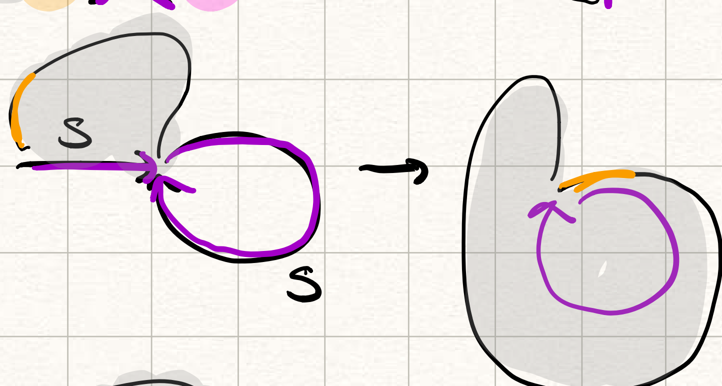
vK's lemma If $w \in F(S)$, then $w = 1$ in $G = \langle S \mid R \rangle \iff \exists$ a reduced vK-diag w/ boundary word w.

Pf: ($\Leftarrow$) If boundary word not reduced, then $\exists$ subpath of ∂D like the following:

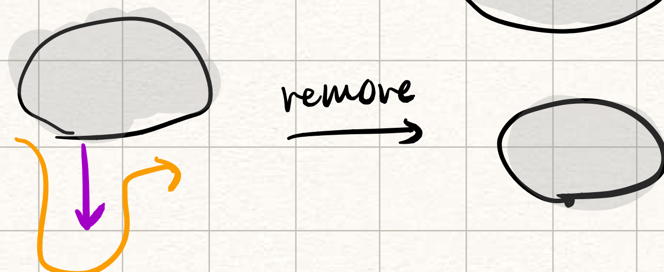
i)

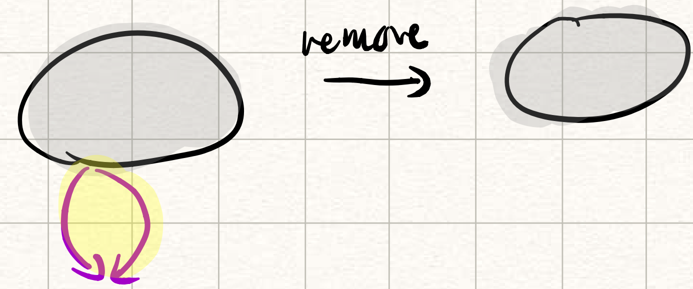


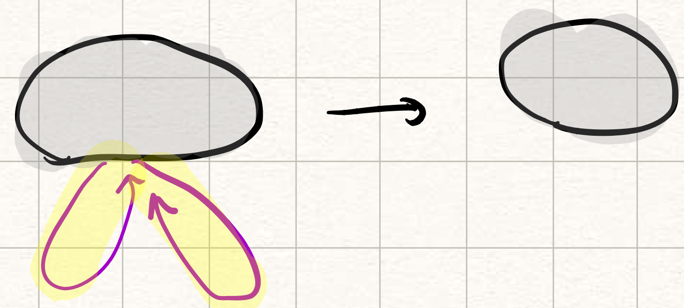
ii)



iii)



iv) 

v) 

Q: How do we know when we got all the cases?

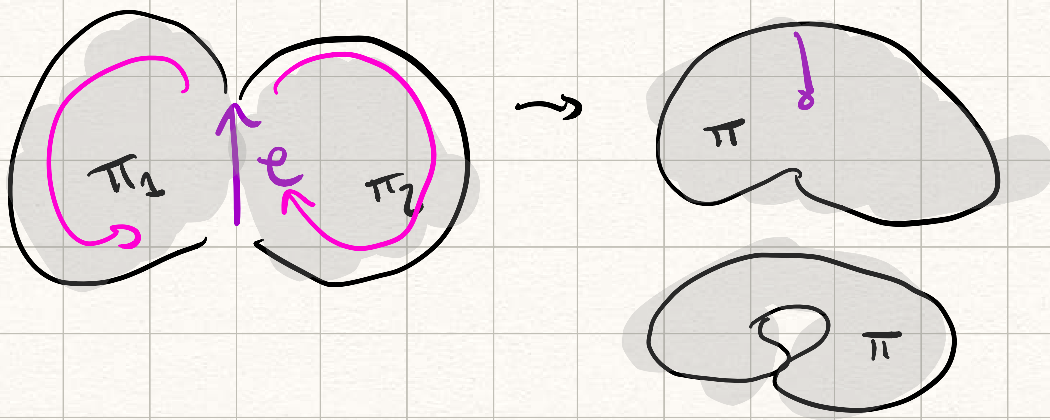
A: check for e_1, e_2, origin o & terminal t:

$\xrightarrow{1}$ $\xleftarrow{2}$

$o(e_1) = \text{ or } \neq o(e_2), t(e_2)$

$t(e_1) = \text{ or } \neq o(e_2), t(e_2)$ ..

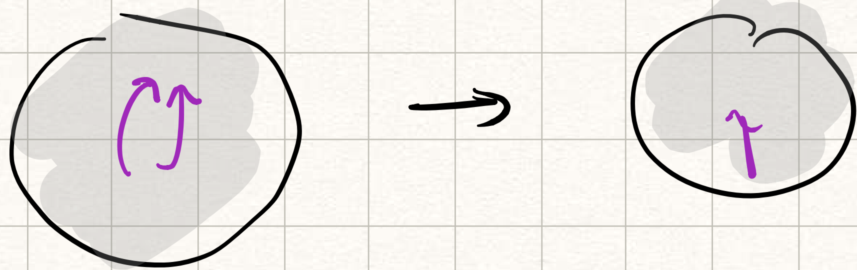
To make D reduced, (i.e. no inverse faces): suppose you have faces π_1 and π_2 in reduction position at edge e. Remove e to get face π w/ freely trivial boundary word.



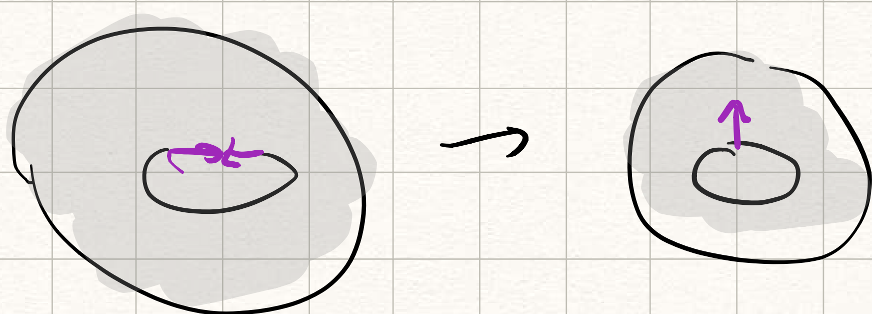
<u>Problem</u>: π might not be nice.

+ how do we get rid of π?

if $|\partial D| \le 2$,



if $|\partial D| > 2$, there is a subpath labeled ss^{-1} (or $s^{-1}s$)

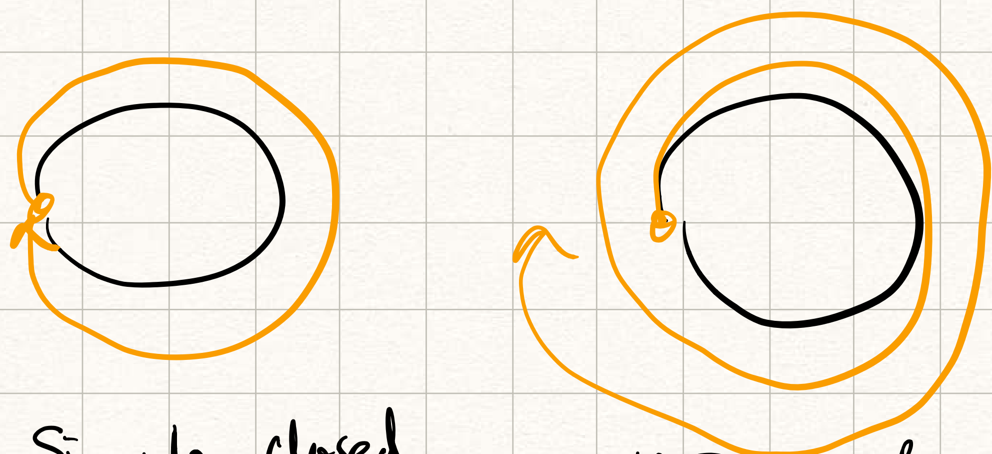


Def: Let D be VK-diag. An arc a is a path which all vertices have degree 2, except possibly endpt.;

the path is simple OR simple closed.

(every vertex visited at most once)

(closed path, every vertex visited at most once except maybe basepoint.)



Simple closed

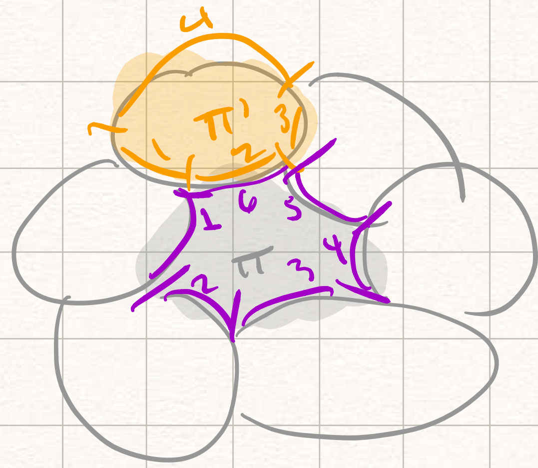
NOT simple closed.

Def:

Let π be a face. The degree of π, denoted $d(\pi)$, is the least number of arcs necessary whose concat. is the boundary path of π.

ex:

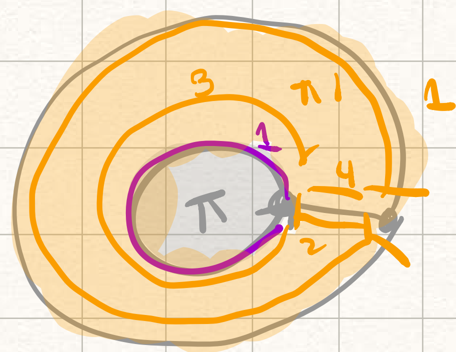
i)



$d(\pi) = 6$

$d(\pi') = 4$

ii)



$d(\pi) = 1$

$d(\pi') = 4$

Def:

If $d = d(\pi)$, and a_i arcs w/ $a_1 \dots a_d$ a boundary path of π, then $i(\pi)$ is the number of interior arcs, and $e(\pi)$ is the number of exterior arcs, where interior means not contained on ∂D.

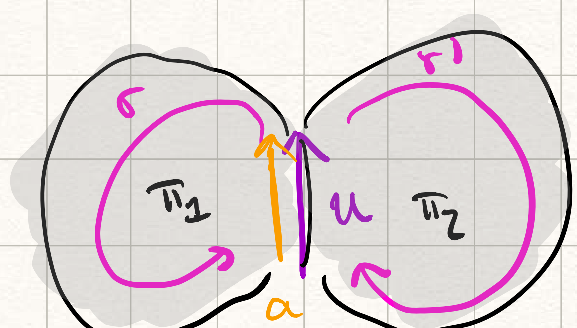
ex: in previous, i) $i(\pi) = 6$

$e(\pi') = 1$

ii) $i(\pi') = \underline{3}$.

Recall: $u \in F(S)$ is a piece if $\exists r, r'$ $r \neq r'$ in $\bar{R}$ st $r \underset{S'}{=} uv$, $r' \underset{S'}{=} uv'$.

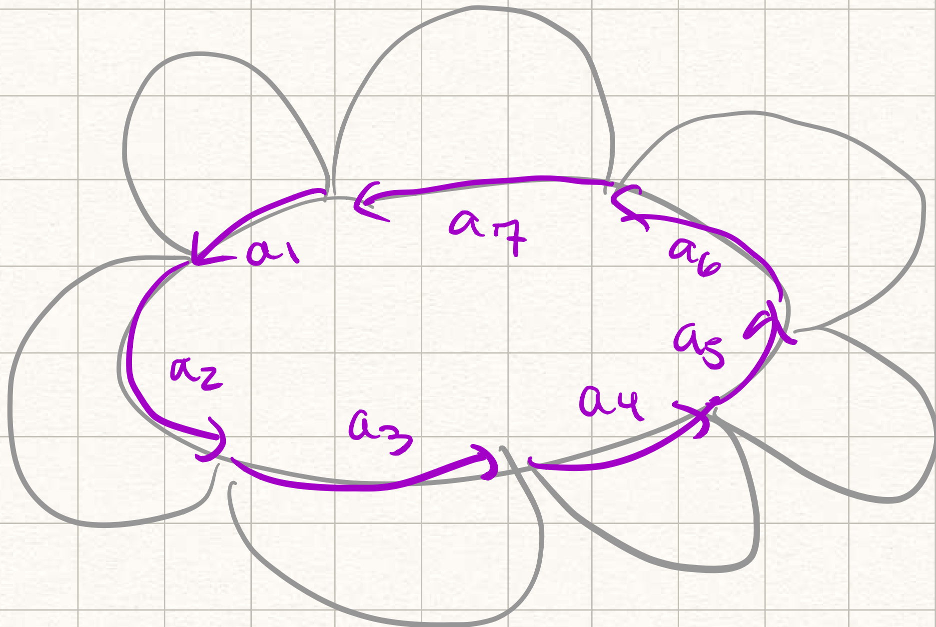
for $v, v' \in F(S)$.



Because D is reduced, $r \neq r'$.
Both r and r' start w/ $\ell(u)$ {label of path a}
and $r, r' \in \bar{R}$.

Cor: If $\langle S | R \rangle$ is a $C(p)$ presentation and D is a reduced diagram over $\langle S | R \rangle$, then every interior face π has $i(p) \geq p$.

$e(\pi) = 0$



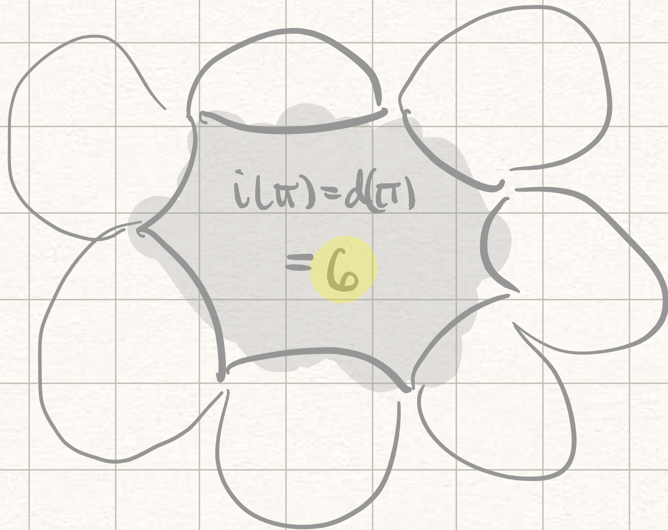
$\ell(a_i)$ is a piece, $\ell(a_1)\cdots\ell(a_7) \in \bar{R}$.
$C(p)$ says: no $r \in \bar{R}$ is a product of fewer than p pieces
$\Rightarrow i(\pi) \geq p$.

Obs: Suppose $\langle S \mid R \rangle$ is a pres. every $r \in R$ is cyclically
(#) reduced. Suppose D is reduced vK-diag. over $\langle S \mid R \rangle$. The the label of any interior arc is a piece.

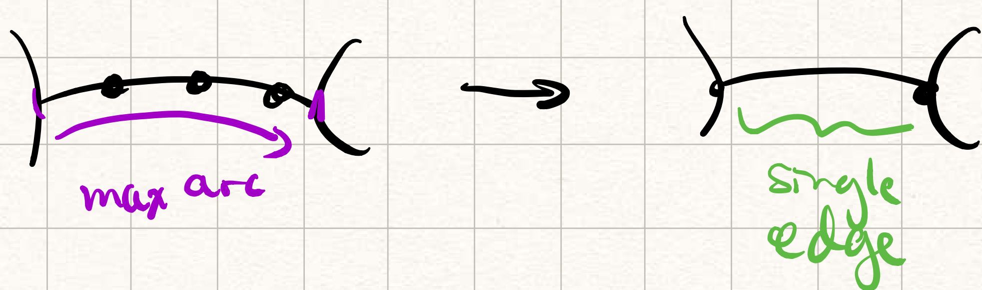
means that it intersects w/ something!

Def: A (vK) diagram D is a $(3,p)$ diag if every interior face π has $d(\pi) = i(\pi) \geq p$

ex : (3, 6) diag.:



<u>Def</u>: If D is a diag. the **consolidation** of D replaces any max. arc by a single edge.



<u>Point</u>: Consolidating does not change the degree of faces.

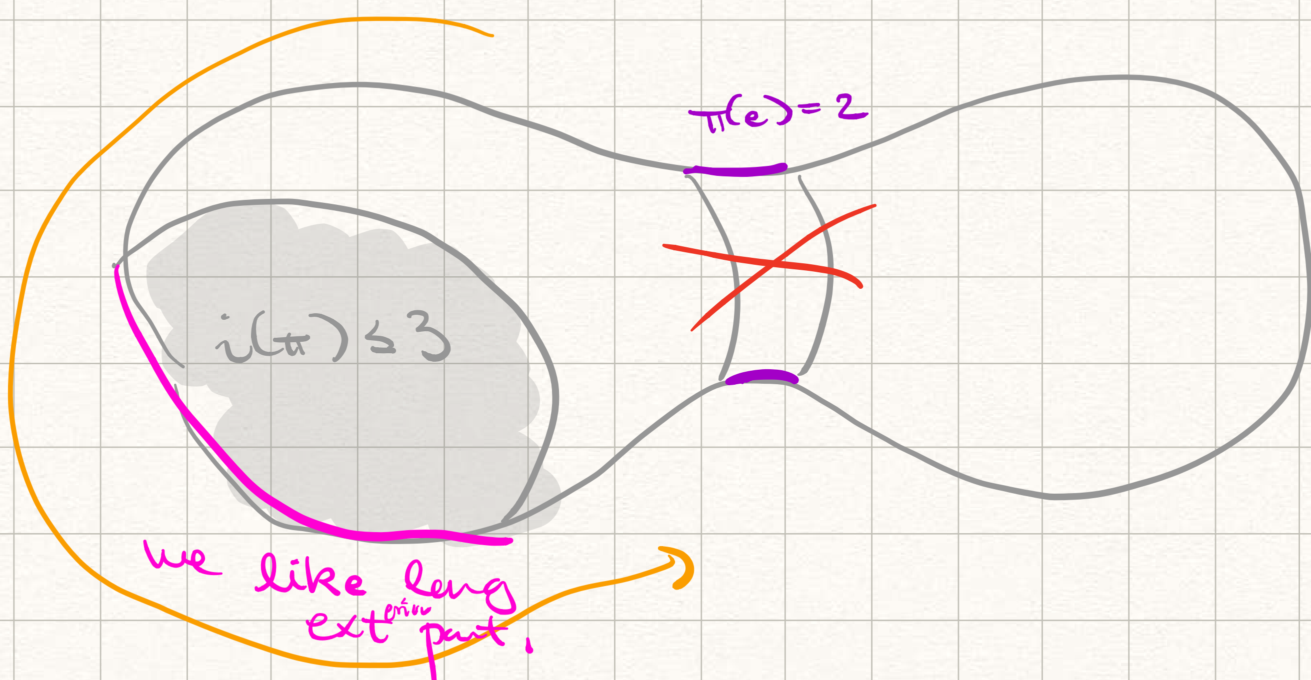
(The 3 comes from the fact that in the consolidation, interior faces have vertices of valence 3).

Lemma: If D is reduced over a $C(p)$ presentation, then $(3,p)$-diag.

Pf: No int. vert. of deg 1, because relators are cycl. reduced (as we did at beginning of lecture).
by (*) cycl. reduced $\Rightarrow i(\pi) \geq p$ for interior faces.

(*) **Thm** (Combinatorial Greendlinger's lem)

Let D be a $(3,6)$ diag. w/ at least 2 faces & no vertex of deg 1.
if cycl. red, no int. vert of deg 1 already, so this assumpt on exterior.
Then D has at least two faces, π_1 & π_2 w/ $e(\pi_1)=1$,
$e(\pi_2)=1$,
and $i(\pi_1) \leq 3$, $\pi(\pi_2) \leq 3$.



Thm: (Euler charact formula)

Let Γ be a finite non-empty connected graph embedded in $\mathbb{R}^2$, then

$$|V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)| = 1.$$

Thm: (Curvature formula)

Let D be non-empty (allows us to use Euler) vk-diag st every edge is contained in a face and no vert. has degree exactly 2.

Then:

$$6 = 2\underbrace{\sum_{v \in V(D)} (3 - d(v))}_{\leq 0} + \sum_{\pi \in F(D)} (6 - 2e(\pi) - i(\pi))$$

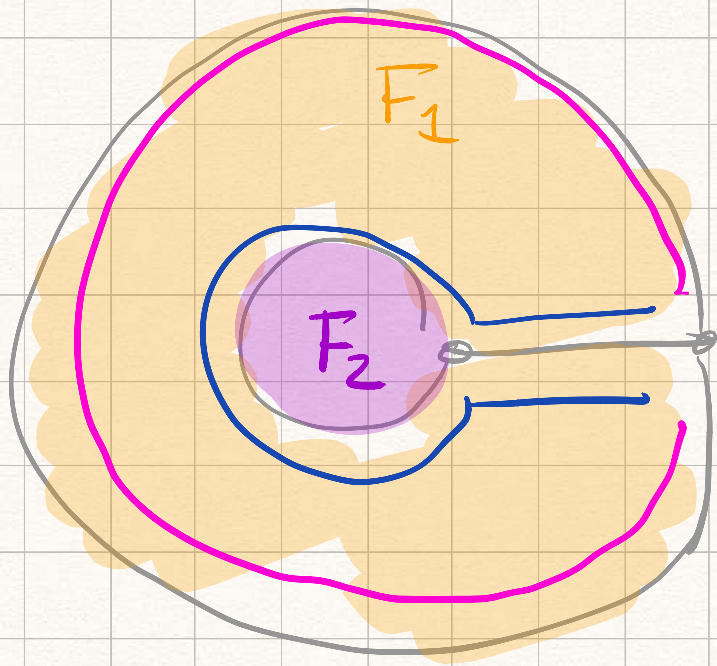
Pf: We have that $i(\pi)$ (resp. $e(\pi)$) is the number of interior (ext.) edges of π, counted w/ multiplicity in the boundary path of π, b/c arcs are edges.

ex:

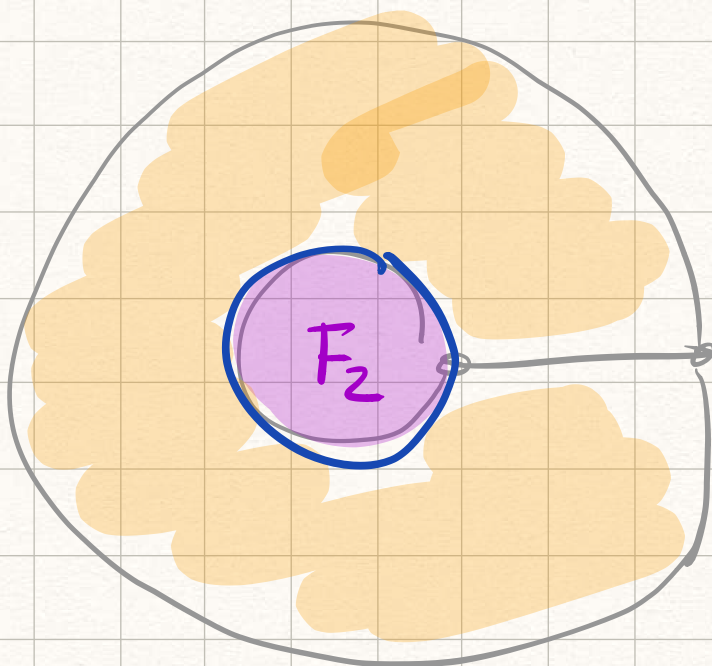
for F_1

— interior

— ext



for F_2

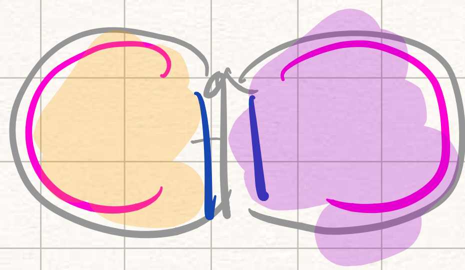


<u>Obs</u>: each interior edge is counted twice, & ext once.

Thus:

$$2|E(D)| = \sum_{\pi \in F(D)} \left(2e(\pi) + i(\pi)\right) \quad (1)$$

also:



and

$$\sum_{v \in V(D)} d(v) = 2|E(D)|$$

basic graph theory fact.

(Euler)

$$\begin{aligned} 6 &= 6|V(D)| - 6|E(D)| + 6|F(D)| \\ &= \left(6|V(D)| - 4|E(D)|\right) + \left(6|F(D)| - 2|E(D)|\right) \\ &= 2\sum_{v \in V(D)} (3 - d(v)) + \underbrace{\sum_{\pi \in F(D)}}_{\text{summing over } |F(D)|} \left(6 - 2e(\pi) - i(\pi)\right) \end{aligned}$$

(1)

$\square$

Now we have enough to show ☆.

Q: Why do we need 2 faces?

A b/c otw you have

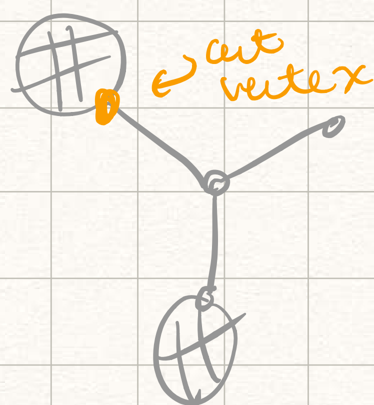
⇒ can't handle ~~this~~ empty ~~thing~~.

☆ Pf of Greendlingers :

i) Assume D has no cut vertices.

{a vertex whose removal disconnects diagram}

Then every edge is contained in a face.



- Replace D by its consolidation (can because at least 2 faces) (allows us to satisfy cond of curv. formula.)

doesn't change degree

use Euler → then get ineq.

- Apply the curvature formula. The only positive contribution comes from faces π w/: $e(\pi) = 1$ and $i(\pi) \le 3$ or $e(\pi) = 2$ and $i(\pi) \le 1$. ①

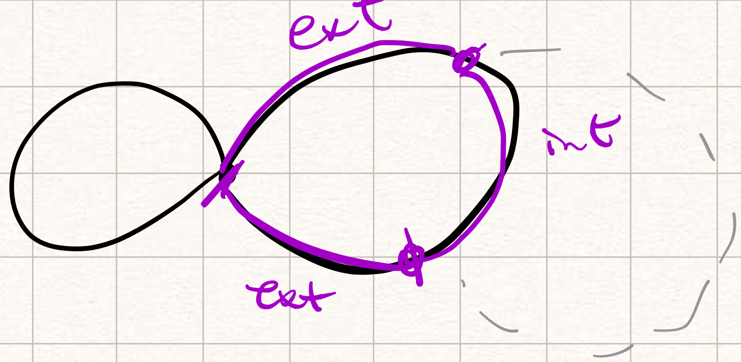
we have no vert. of deg 1 in thm statement

exc: check.

Suppose π has $e(\pi) = 2$, and $i(\pi) \le 1$.
Then there is a cut vertex in the intersection of 2 max. exterior arcs in π ⟹ π cannot exists!

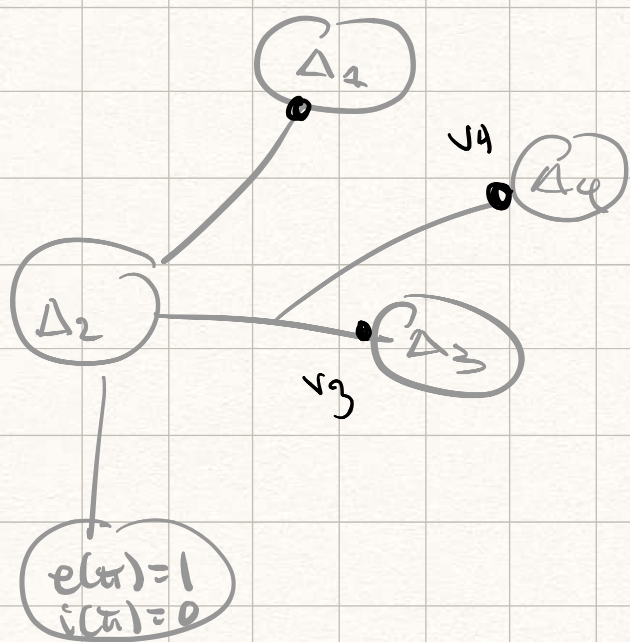
contra

so there are ≥ 2 faces π which satisfy ①



ii) General case,

Now suppose that D has cut vertices. Then D is a tree diag. T $\Delta_1, \ldots, \Delta_n$ st no Δ_k has cut vertices,



If Δ_i is a leaf of T, then Δ_i intersects exactly one vertex v_i of $D \setminus \Delta_i$. So Δ_i has at least one face as in ① with degrees counted in D.

There is at least 2 leaves

$\Rightarrow$ Claim follows.

$\square$