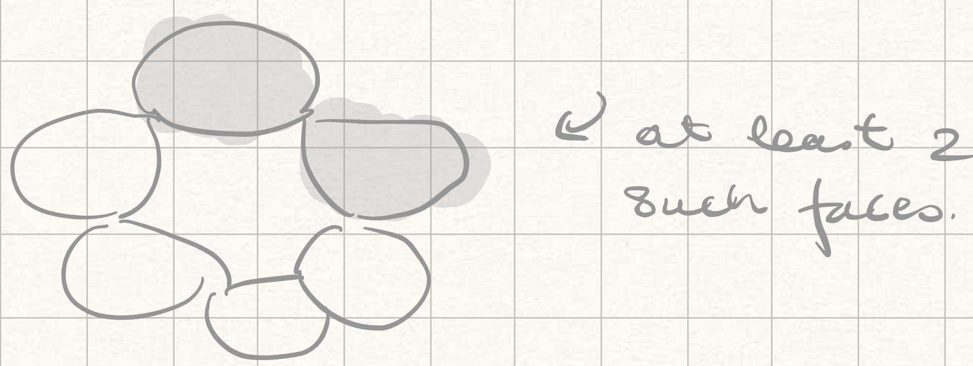
Thm: Let D be a $(3,6)$ diag w/ at least 2 faces, no vertices of deg 1. Then D has at least 2 faces π_1, π_2 w/ $e(\pi_1) = e(\pi_2) = 1$, $i(\pi_1), i(\pi_2) \leq 3$.

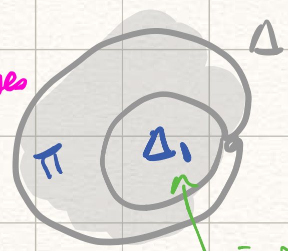


Cor: Let D be a $(3,6)$ diag. Then every face is simply connected. (simply closed boundary path)

Pf: Take the face π which is NOT simply connected. Then π encloses some subdiagrams of D, $\Delta_1, \dots, \Delta_k$ st $\pi_1 \cup \underbrace{\Delta_1 \cup \dots \cup \Delta_k}_{\text{inside faces of } \pi}$ is a by def, has no int vertices of deg 1, so we're ok.

(3,6) diag with more than 1 face, but only 1 boundary face.

π encloses at least 1 face

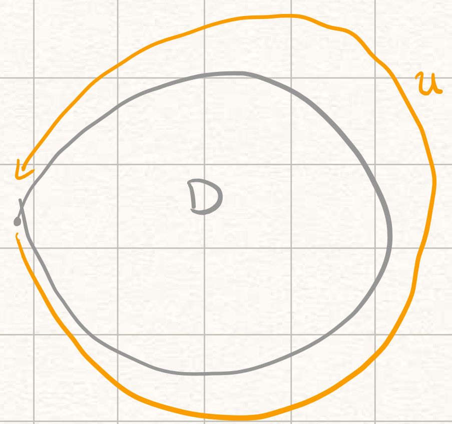


inside face of π

Greendlinger says needs at least 2 bd face.

Thm 1: Let $\langle S \mid R \rangle = G$ be a $C(6)$ pres, u non-empty proper subword of $r \in \bar{R}$. Then $u \neq_G 1$.

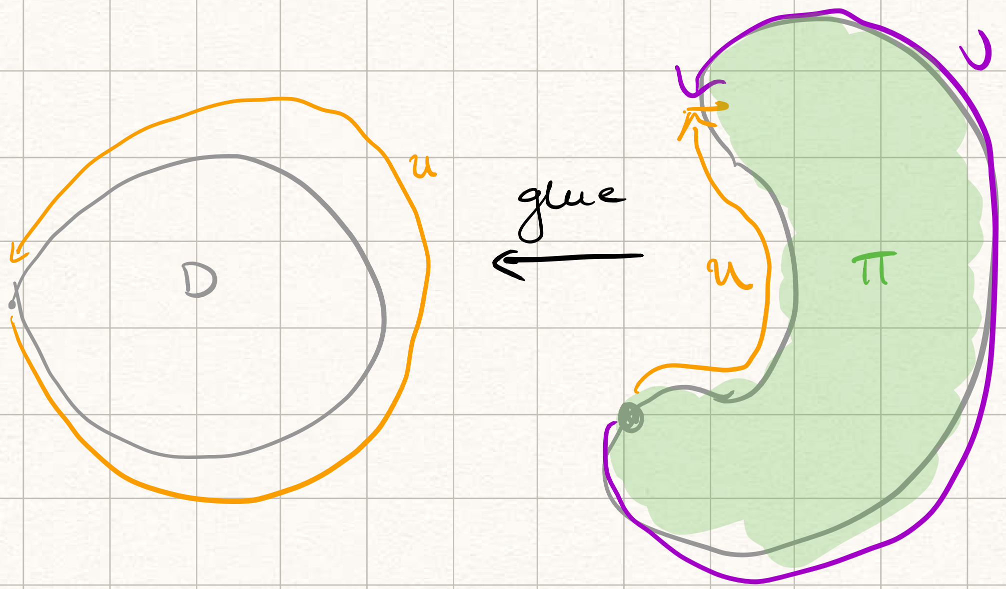
Pf: Assume $u =_G 1$, let D be reduced vk-diag over $\langle S \mid R \rangle$ w/ boundary word u.



Assume that among all choices for u, and subsequently D, the number of faces of D is minimal.

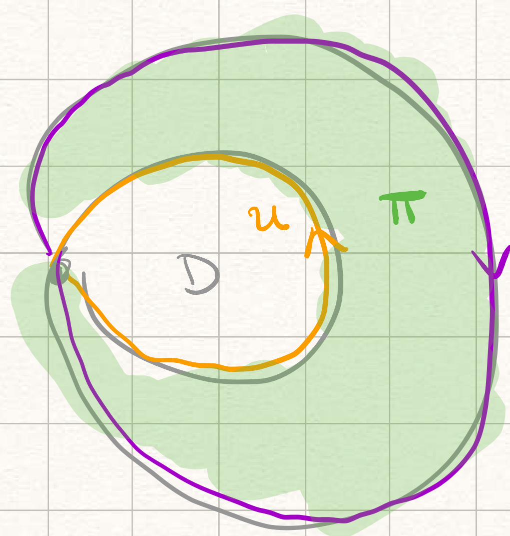
Observe: D has at least 1 face b/c u is non-trivial in the free grp. otw D would be a diagram over the presentation $\langle S \mid \emptyset \rangle$ $\Rightarrow u \underset{F(S)}{=} 1 \rightsquigarrow u$ empty. contra.

Write: $r = uv^{-1}$ for some $v \in F(S)$.
Let π be a new face w/ boundary word r.



$\parallel$

$\Delta =$



Claim: Δ is reduced. Suppose not. Then, as in the proof of vK lemma, we can reduce the number of faces of Δ by at least 2, preserving the boundary word v.

$\rightsquigarrow$ let Δ' be such a reduced diagram.

$$\Rightarrow |F(\Delta')| \leq |F(D)| + 1 - 2$$
$$= |F(D)| - 1$$
$$< |F(D)|$$

Also, v is a non-empty proper subword of $r \in \bar{R}$. $\leftarrow$ contradiction of minimality & we're done.

$\rightsquigarrow$ So Δ is reduced. Therefore Δ is a $(3,6)$ diagram w/ a non-simply connected face $\rightsquigarrow$ contradiction of corollary. (Δ).

because the interior has u_1 of deg 6.

Note: we use u proper $\Rightarrow v$ non-empty.

$\square$

<u>Rk</u>: Consider $Cay(G,S)$, $G=\langle S\rangle$.

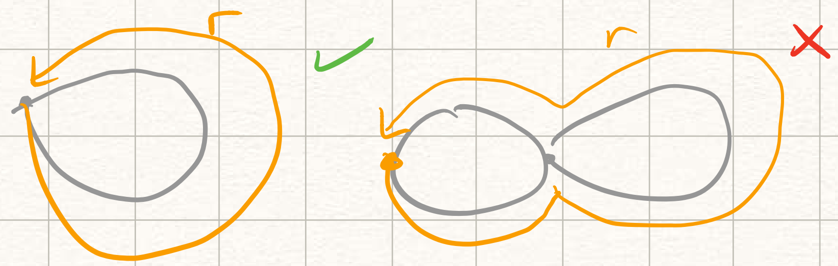
Set of labels of closed paths in $Cay(G,S)$ is <u>exactly</u> the set of words in $S\cup S^{-1}$ (possibly non-reduced) representing 1_G, because if p is a path starting at $g\in G$, the endpt is $g\cdot l(p)$

$$g \xrightarrow{s} gs$$

so p is closed $\iff g =_G g\cdot l(p)$
$\iff 1 =_G l(p)$.

If $G=\langle S\mid R\rangle$, then Thm 1 says: no non-empty closed subpath of a relator labels a closed path in $Cay(G,S)$. Hence any path in $Cay(G,S)$ labeled by a relator is closed and has no non-empty closed subpath $\rightarrow$ hence it is simply closed.

In other words, any relator labels an embedded cycle graph.



□

Note: This is why we get away by saying "hyperbolic grp" & showing a pic of a hyperbolic tesselation
⇝ the Cay graph is REALLY just a bunch of relators (simply connect faces) glued together!

Dehn's algorithm for WP

origins of small cancellation!

Def: (Metric small cancellation condition)

Pt: pieces small if: – takes lots of them to make an $r \in R$

OR: actual fraction of relator

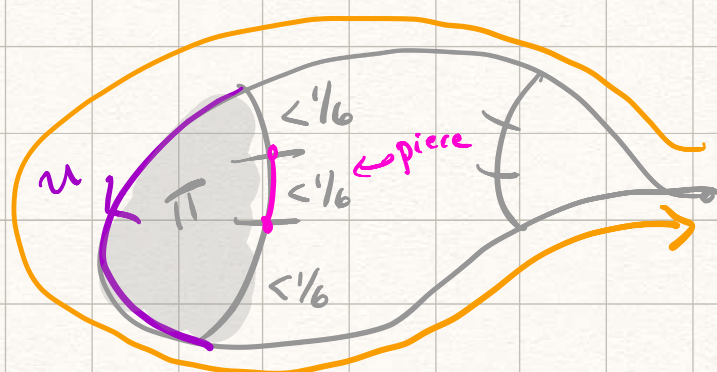
this one (stronger)

Let $\langle S \mid R \rangle$ be a presentation, $\lambda > 0$.
Then $\langle S \mid R \rangle$ satisfies the $C'(\lambda)$ cond.
if :
- R is concise
- every $r \in R$ is cyclic. red.
- whenever a piece u is a subword of $r \in \bar{R}$, then $|u| < \lambda |r|$.

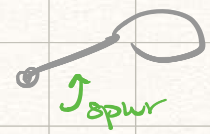
<u>Notice</u>: If $\langle S \mid R \rangle$ is $C'(\frac{1}{p})$, then it is $C(p+1)$. ↑ because $|u| < \frac{1}{p}|r|$ $\Rightarrow |r| > p|u|$ $\Rightarrow C(p+1)$.

ex: $\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$
pieces have len ≤ 1
$\Rightarrow C'(1/7)$ is satisfied.

<u>Thm</u>: (Greenlinger's lemma)
Let $\langle S \mid R \rangle = G$ be a $C'(1/6)$-pres.
and let $w \in F(S)$ non-empty and rep'ing 1_G. Then w contains a subword u st u is a subword of some $r \in \bar{R}$ st $|u| > |r|/2$.

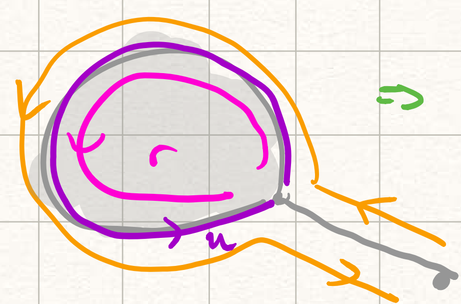


Pf: A spur in a diagram is an arc w/ a terminal vertex of degree 1.



(Pb: can't have 'em around except at beginning b/c w is reduced)

Because w is reduced, D has at most 1 spur. If D has exactly one face, then w has a subword as in the claim.



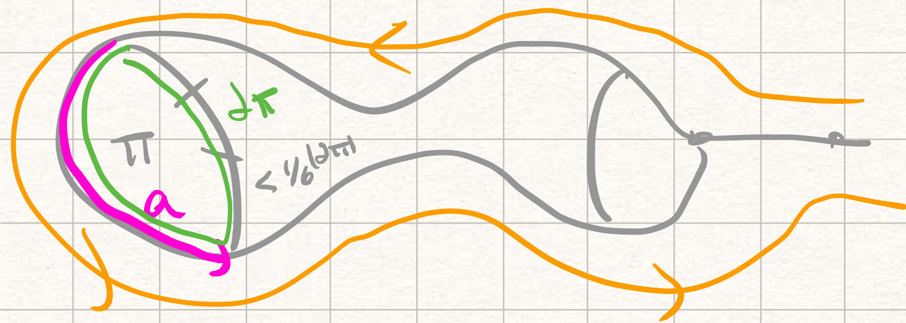
$\Rightarrow u = r \in \bar{R}$.

If D has more than 1 face, remove the spur to get a diagram D' to which combinatorial Greenlingers applies $\rightsquigarrow$ D' has at least 2 faces w/ ext degree 1, int degree ≤ 3. If at least one of the two faces, say Π, has its boundary arc a a subpath of ∂D

(b/c $C'(1/6)$ $\Rightarrow C(6)$ $\Rightarrow (3,6)$.)

looking at D here b/c we want

w, not conjugate of w.



Then $|a| > |\partial\pi| - 3\frac{|\partial\pi|}{6}$,
so w contains more than $1/2$ of the boundary word of π.

Dehn's algo: Let $\langle S | R \rangle$ finite $C'(1/6)$ pres. Let $w \in F(S)$.

Step 1: If w contains subword u st $\exists\, r \in \bar{R}$ w/ $r \equiv uv^{-1}$ w/ $|u| > |v|$, replace u by v and freely reduce. Restart.

Step 2: If w is empty, return "trivial", terminate.

Step 3: Else, return "non-trivial", terminate.

Obs: Step 1 does not change the elmt of G rep'd by w b/c $u =_G v$. It also strictly reduces the len of w

⇝ always terminate.

Greenlingers tells us that if w trivial, we will find such a u, otw w NOT trivial.

Q: Why finite pres?

A: B/c at step 1, check against $\overline{R}$, which needs to be a finite collection. Note: the real requirement is for u to be recog as a subword

Thm: Let G be given by a finite $C'(1/6)$ pres. $\langle S \mid R \rangle$, then Dehn's algorithm solves the WP.

Q: Are there $C'(1/6)$ grps w/ unsolvable WP?

A: Yes. We are going to show there are uncountably many of them (but only countably many Dehn algos).

Pf: Let $\langle a, b \mid r_7, r_8, \ldots \rangle$ be an infinite $C'(1/6)$ pres.

e.g. $r_n = (a^n b^n)^n$.

For $I \subseteq \mathbb{N}_{\geq 7}$, let $R_I = \{r_i : i \in I\}$, then $\{R_I : I \subseteq \mathbb{N}_{\geq 7}\}$ is uncountable.

there are uncountably many subsets of $\mathbb{N}$.

If $n \notin I$, then $r_n \neq 1_G$ in $\langle S \mid R_I \rangle$, because r_n does not contain more than $\frac{1}{2} r_i$ for $i \in I$; $\Rightarrow$ so by Greendlinger $r_n = 1_G$ cannot be.

because any common subword of r_n & r_i is a piece in the pres $\langle S \mid r_7, r_8, r_9, \ldots \rangle$

& G is $C'(1/6)$, so each piece $< 1/6 |r_i|$.

$\Rightarrow r_n \notin \langle\langle R_I \rangle\rangle$.

$\Rightarrow I \neq J$, then $\langle\langle R_I \rangle\rangle \neq \langle\langle R_J \rangle\rangle$.

the map $I \mapsto \langle\langle R_I \rangle\rangle$ is injective.

$$\left\{ \begin{array}{l} \text{Lemma: If } G \text{ is a countable grp, there are at most countably many epimorphisms } \phi: F(a,b) \to G. \\ \text{Pf: } \phi \text{ determined by } (\phi(a), \phi(b)) \in G \times G. \quad \square \end{array} \right\}$$

$\Rightarrow$ If G is countable, there are at most countably many normal subgrps $N \leq F(a,b)$ st

$$G \cong F(a,b)/N.$$

$\Rightarrow$ $\langle S | R_I \rangle$ defines uncountably many grps, but there are finitely many Dehn algs, so some of them CANNOT have a solvable WP.