Yesterday:

① Let $N \trianglelefteq F(a,b)$. Then $G = F(a,b)/N$ has soluble WP wrt $\{a,b\}$ if $\exists$ an algo. A in the alphabet $\{a,b\}$ st $\forall w \in F(S)$:

- $A(w) =$ "trivial" $\iff w \in N$
- $A(w) =$ "non-triv" $\iff w \notin N$.

$\Rightarrow$ A determines N uniquely
$\Rightarrow$ if $N \neq N'$, then $A \neq A'$

② But we showed that there exist uncountably many normal subgrps N of $F(a,b)$ (using Green lingers), and only countably many alg. in $\{a,b\}$.

① + ② $\Rightarrow$ there exists $N \leq F(a,b)$ for which there are no algo!

Thm: There exists $N \trianglelefteq F(a,b)$ st $F(a,b)/N = G$ which has unsolvable WP wrt $\{a,b\}$.

exc : Solvability of WP does not depend on the finite generating set.

Graphical Small Cancellation

History : - Dehn Algo (1910 Dehn)
- Classical small cancell. (1949 Tartakovski)
- Graphical s.c. (2003 Gromov / Ollivier '06, Speaker '15)

☆ Q : Given Γ, how to produce a fg grp whose Cayley graph contains Γ as embedded subgraph?

Note : if NOT fg, trivial b/c can make Cayley graph w/ complete graph w/ ∞ - vertices
$\Rightarrow$ contains everything.

Applications : • grps w/ embedded expanders (certain infinite graphs)
$\longrightarrow$ looking at asympt. geom.

- disproved Baum-Connes conjectures w/ coeffs

- first grps w/ non-exact reduced C^*-alg, not coarsely embedded into Hilbert space.

Def: Let Γ be a S-labeled graph. We say Γ is <u>reduced</u> if it has no subgraph of the form

$$\xrightarrow{s}\xleftarrow{s} \qquad \xleftarrow{s}\xrightarrow{s}$$

The grp defined by Γ a reduced S labeled graph is

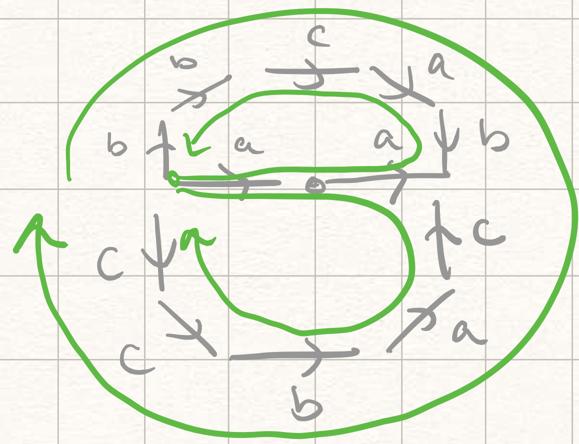
R_Γ

$G(\Gamma) = \langle S \mid$ words read on simple closed paths in $\Gamma \rangle$

injective copy of circle S^1.

<u>Obs</u>: every $r \in R_\Gamma$ is cyclically reduced.

ex: $S = \{a, b, c\}$

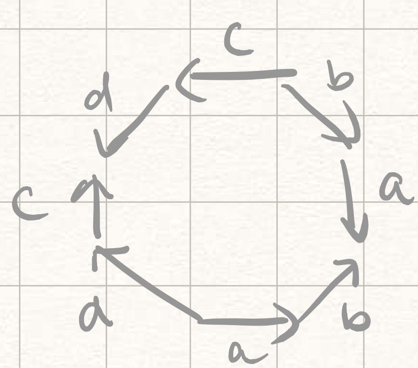


$$R_\Gamma = \{ a^2 b^{-1} a^{-1} c^{-1} b^{-1} b^{-1},$$
$$a^2 c^{-1} a^{-1} b^{-1} c^{-2},$$
$$b^2 c a b c^{-1} a^{-1} b^{-1} c^{-2},$$
+ all cyclic conj, inverses}

we have $R_\Gamma = \overline{R_\Gamma}$.

ex: Let $R = F(S)$ be a set of cycl. red. words. Let Γ be a disjt union of cycle graphs labeled by the elmt of R, e.g.

$$\langle a, b, c, d \mid a b a^{-1} b^{-1} c d c^{-1} d^{-1} \rangle$$



Then $R_\Gamma = \bar{R}$. In particular, if $G = \langle S \mid R \rangle$,

$G(\Gamma) = \langle S \mid R_\Gamma \rangle = \langle S \mid \bar{R} \rangle = G$.

Lemma 1: Let Γ be a reduced S-labeled graph, let $\Gamma_0 \subseteq \Gamma$ be a connect. component. Let $v \in V(\Gamma_0)$, $g \in G(\Gamma)$. Then $\exists$ a unique label-preserving hom f,

$f : \Gamma_0 \to \mathrm{Cay}(G(\Gamma), S)$ w/

$f(v) = g$.

what they are saying is that you can answer *Q, and furthermore that is bpt indep.

Lemma: Let w be a word in $S \cup S^{-1}$ read on a closed path in Γ. Then $w = 1$ in $G(\Gamma)$.

Pf: If w is read on a simple closed path, then $\underbrace{w \in R_\Gamma}_{\text{by def of } G(\Gamma)}$.

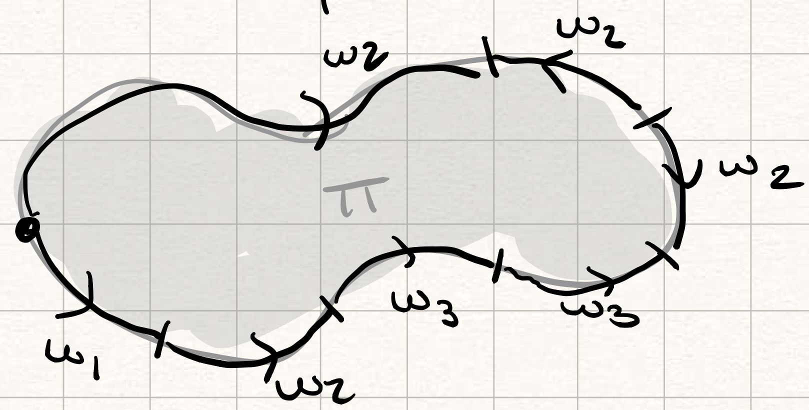
$\Rightarrow w =_{G(\Gamma)} 1$.

If w read on a non-simple closed path (but closed path).

Γ: 

$w = \underbrace{w_1 w_2 w_3^2 w_2^{-1} w_2 w_2^{-1}}_{P}.$

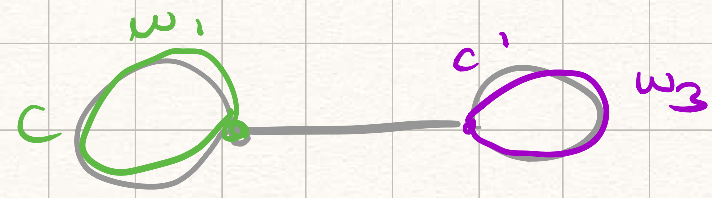
Let Π be a face w/ boundary word w.

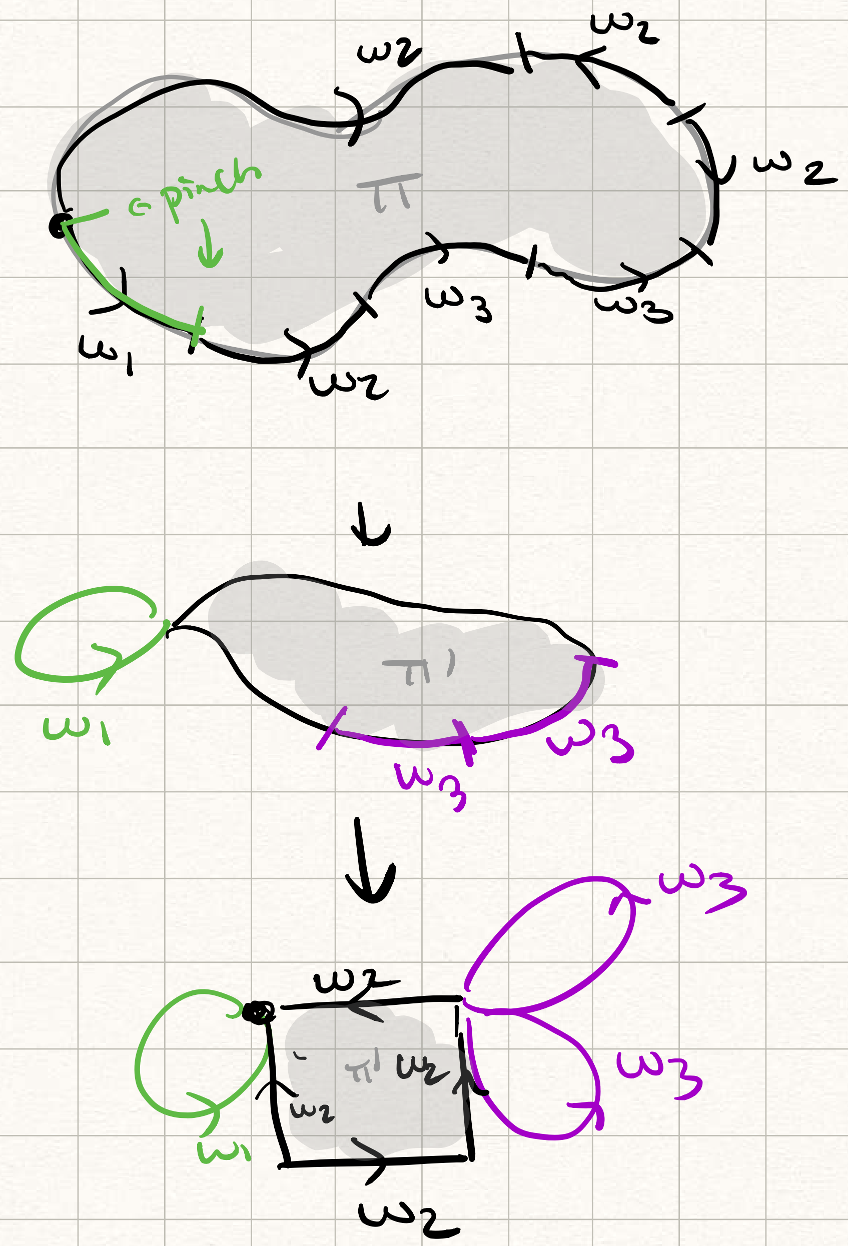


$\rightsquigarrow$ We will turn Π into a diag over $\langle S | R \rangle$.

If P has a subpath that is
- simple closed or
- of the form ee^{-1} or $e^{-1}e$ for some edge.

If p has a subpath c that is simple closed, pinch together endpts of corresponding subpath of $\partial\Pi$.





Remove subpaths of $\partial\pi$ w/ labels ss^{-1} or $s^{-1}s$ as in vK lemma.
Do this until π is removed or $\ell(\partial\pi)$ is in R_T.

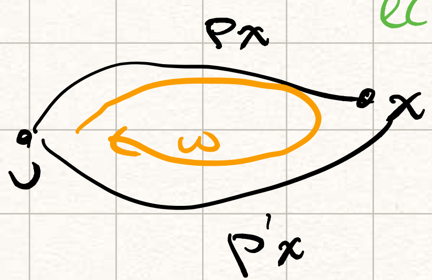
Q: Why does it work?

A: B/c we are only pinching out at closed paths so the path p remains is still closed $\Rightarrow$ so the path p becomes either simple closed (thus in R_p) or freely trivial.

Pf (of Lemma 1):

Let $f: \Gamma_0 \to \mathrm{Cay}(G(\Gamma), S)$ be defined by $x \longmapsto g \cdot \ell(p_x)$ where p_x is a path from v to x. By lemma, this does not depend on the choice of p_x

in ~~the sense that~~ $\ell(p_x)\ell(p'_x)^{-1} = \omega$ by lem,



$\omega =_G 1$ by lemma.

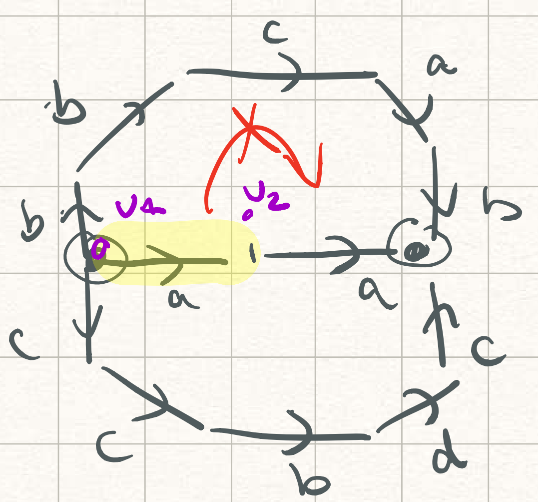
so $\ell(p_x) = \ell(p'_x)$?

If f is label-preserving, this is the only choice.

exc: f gives label preserving hom.

Def (Graphical sc. cond) Let Γ be S-label graph. A $\underline{\Gamma\text{-piece}}$ is a word w for which there exists 2 vertices v_1, v_2 starting from which one can read w st $\forall \phi$ label-preserving aut of Γ, we have $v_1 \neq \phi(v_2)$. ← label + orientation

ex:



there are no non-trivial aut → look at 2 vertices ⊙ + orientation.

pieces: a can be read from v_1, v_2.
total: $\{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$.

⇝ C(7) condition.

Obs: a^2 was a piece in classical case, so graphical cond more general.
in the sense that piece in Graph sc

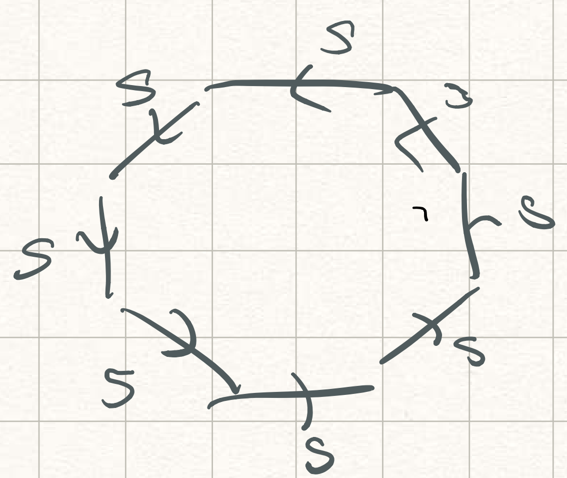
⇒ piece in classical sc.

non-empty

ex2 : Cay(G, S) has no non-empty pieces b/c can always get from one vertex to another by left mult, which is an automorphism.

⇝ Satisfies every C(p) cond.

ex3 : $\langle s \mid s^8 \rangle$



Rotation is aut ⇒ pieces are empty. (would have pieces w/o aut cond).

⇒ Γ satisfies every C(p)

Obs : $\langle s \mid s^8 \rangle$ satisfies every classical C(p) cond.

exc : $G = G(\Gamma)$.

Rk: There exists a finite $C(p)$ labeled graph Γ st $G(\Gamma)$ is NOT $C'(1/6)$ - classical s.c grp (or isomorphic to any).

Thm: Let Γ be $C(6)$-labeled graph, let Γ_0 be a connect. comp. Then, every label-preserving hom from $\Gamma_0 \to Cay(G, S)$ is injective.

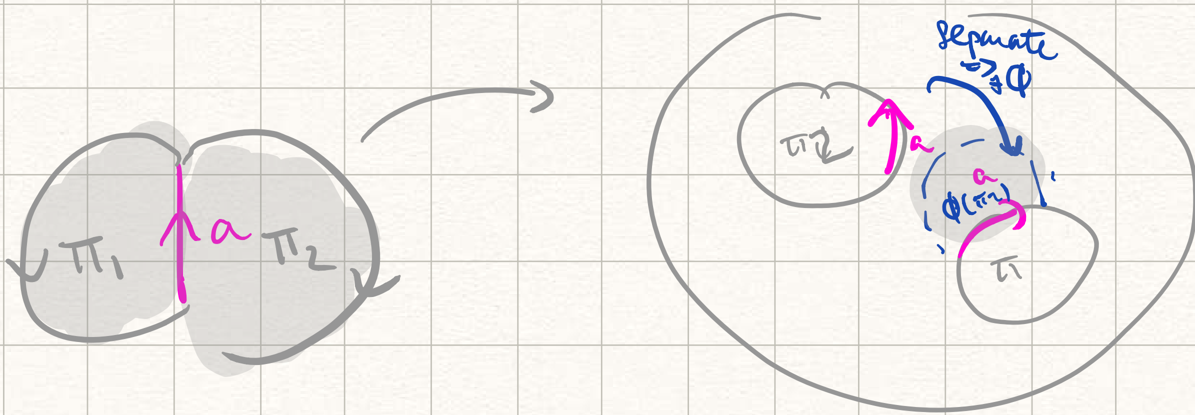
(analogue of Thm 1).

Thm (Graphical vK lemma)

Let Γ $C(6)$-labeled graph, $w = 1$ in $G(\Gamma)$. Then there exists an reduced vK-diag over $\langle S \mid R_\Gamma \rangle$ w/ bound. word w st the label of every interior arc is a Γ-piece.

Cor: If Γ is $C(p)$ for $p \geq 6$, then D is a $(3,6)$ diag.

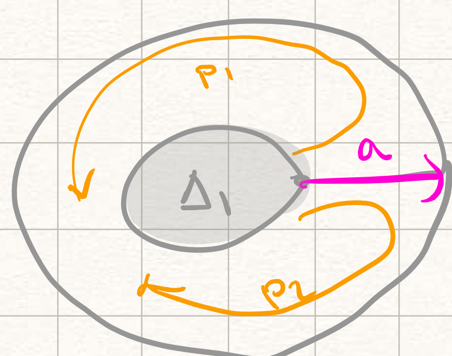
Idea: Suppose a in $\pi_1 \cap \pi_2$ not a piece.



Pf: (of vK lem)

Let D be vK over $\langle S \mid R_\Gamma \rangle$ w/ bound. word w. Assume that among the choices for D, the number of edges is minimal (for contra).

Let a interior arc st $\ell(a)$ not a Γ-piece. Realize a as the common initial subpath of the positive boundary path p_1 of Π_1, and the neg. boundary path p_2 of Π_2, for faces Π_1, Π_2.

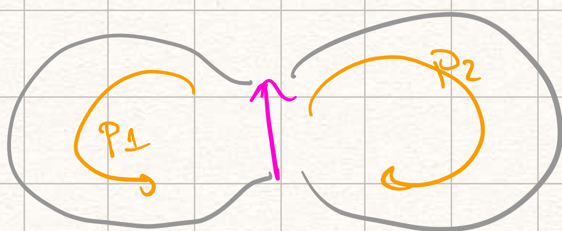


Suppose $\pi_1 = \pi_2$. Then π_1 is not simply connected $\Rightarrow$ it encloses some subdiag. $\Delta_1, \Delta_2, \dots \Delta_k$.

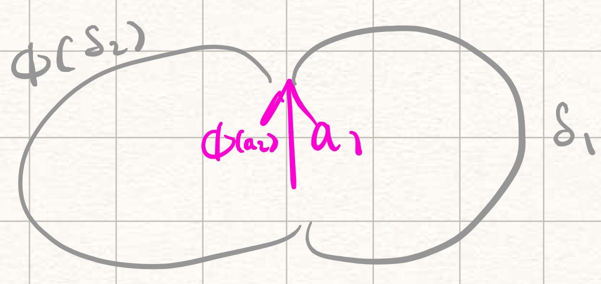
why?

Then $\Delta = \pi_1 \cup \Delta_1 \cup \dots \cup \Delta_k$ is no $(3,6)$ diag. This means $\exists$ at least 1 int. face of Δ w/ arc in its boundary whose label is not a Γ-piece.

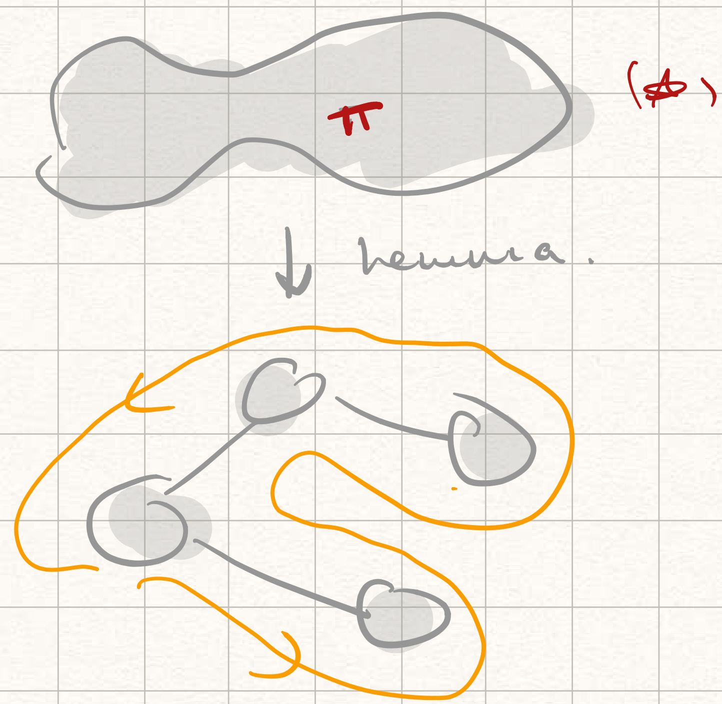
So by going to innermost config, we can assume $\pi_1 \neq \pi_2$. There simple closed paths γ_1, γ_2 w/ same labels as p_1, p_2. Both γ_i start w/ subpaths $\ell(a_i) = \ell(a)$. Write $\gamma_i = a_i \delta_i$.



Since $\ell(a)$ not piece $\Rightarrow \exists\, \phi$ an aut st $\phi(a_2) = a_1 \Rightarrow \delta_1^{-1} \phi(\partial_2^{-1})$ is a closed path in Γ.



Remove the arc a from D to get a new face π w/ boundary word labeling a closed path in Γ. Replace π by the diag. over $\langle S \mid R_\Gamma \rangle$ constructed in Lemma after Lemma 1,



in order to obtain D', a diag over $\langle S \mid R_\Gamma \rangle$ w/ bdry word w/ fewer edges
boundary

$\Rightarrow$ Contra $\square$.

*: <u>Note</u>: May get non-nice cases if π is NOT simply connected, but can resolve as in vK-lemma.

Exercises

1) exc : Solvability of WP does not depend on the finite generating set.

Suppose $G \cong \langle S | R \rangle \cong \langle S' | R' \rangle$ both fg presentations, and that $\langle S | R \rangle$ has solvable WP w/ algo A.

Lemma: If S' finite, we can algorithmically map $\langle S' \rangle \to \langle S \rangle$.

Pf: Take $\{y_1, \ldots, y_n\} = S'$.
We know that each y_i corresponds to a word $w(y_i) \in S^*$. Let $\{w_1, \ldots, w_n\}$ be these words.
We want to algorithmically find their description in terms of gens S.

IDK!

We know that $\exists w^*$ st $|w^*| = \max |w_i|$.

× look it up!

Outstanding Q

I still am NOT clear on what is a label-preserving hom..