# Polycyclic Groups (PART 2)

Recall:

- Polycyclic group

  $G = G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = 1$

  where the quotients are cyclically generated and $G$ admits a nice polycyclic presentation.

  some exponent

  $x_i^{s_i} = x_{i+1}^{*} \cdots x_n^{*}$

  $x_i^{x_j} = x_{j+1} \cdots x_n^{*}$

  $x_i^{x_j^{-1}} = x_{j+1}^{*} \cdots x_n^{*}$

- GAP $\to$ pc group (built-in); $\to$ pcp group (external)

  both for polycyclic, just different packages

ex: $S_4$

pcgs $x = [(34), (243), (13)(24), (12)(34)]$

$R(x) = (2, 3, 2, 2)$

$x_1^2 = 1$
$x_2^3 = 1$
$x_3^2 = 1$
$x_4^2 = 1$

$x_2^{x_1} = x_2^2$
$x_3^{x_1} = x_4$
$x_3^{x_1} = x_4$
$x_3^{x_2} = x_4$
$x_4^{x_1} = x_3$
$x_4^{x_1} = x_3$
$x_4^{x_2} = x_3 x_4$
$x_4^{x_3} = x_4$

Q: How to do this using buit-in pc group package?

# GAP demo

\# By fp-groups

F:= FreeGroup("a", "b", "c", "d")
AssignGeneratorVariables(F);
rels:= (write down relations)
G:= F/rels

we'll see why we need this

IsConfluent(G);
StructureDescription(G);

H:= PcGroupFpGroup(G);

```
# S4 using pc collector.

# no 1: from an fp presentation:
f := FreeGroup("a", "b", "c", "d");
AssignGeneratorVariables(f);
rels:=[a^2,b^3,c^2,d^2,b^a/b^2,c^a/d,c^b/d,d^a/c,d^b/(c*d), d^c/d];
g := f / rels;
h :=PcGroupFpGroup(g);
StructureDescription(h);
```

we'll also talk about this today

\# By collector

F:= FreeGroup(4);

Relative orders

col:= SingleCollector(F, [2,3,2,2])

SetConjugate(col, 2, 1, F.2^2);

(...)

g:= GroupbyRws(col);

IsConfluent(g);
StructureDescription(g);

```
# no 2: pc collector

f:=FreeGroup(4);
col:=SingleCollector(f,[2,3,2,2]);
SetConjugate(col, 2, 1, f.2^2);
SetConjugate(col, 3, 1, f.4);
SetConjugate(col, 3, 2, f.4);
SetConjugate(col, 4, 1, f.3);
SetConjugate(col, 4, 2, f.3*f.4);
g :=GroupByRws(col);
IsConfluent(g);
StructureDescription(g);
```

$x_2^{x_1} = x_2^2$

ex ($S_4$): $G = Pc\langle x_1, x_2, x_3 \mid x_1^3 = x_3,\ x_2^2 = x_3,\ x_2^{x_1} = x_2 x_3 \rangle$

power exponent

($S$ in polycyclic pres $x_i^{s_i} = x_{i+1}^{a_{i,i+1}} \cdots$)

$S(x) = [3, 2, \infty]$

$R(x) = [3, 2, 1]$

$R(x) \neq S(x)$

$x_2^{x_1} = x_2 x_3$

$\Rightarrow (x_2^2)^{x_1} = x_2^2 x_3^2$

$x_3^{x_1} = x_3^3 \Rightarrow x_3 = 1$

## Consistent presentations

A pc presentation is not consistent (or confluent) if $S(x) \neq R(x)$.

This is equivalent to the fact that $\exists$ an element in $G$ which can be presented in two ways as a normal form.

ex ($S_4$): Consider:

$$\underbrace{x_1 x_2}_{\text{NORMAL FORM}} = x_1 x_1^2 = x_1^2 x_1$$

$$= x_2 x_1 = \underbrace{x_1 x_2 x_3}_{\text{NORMAL FORM}}$$

Let's look at the theory of consistent presentations.

Let $G$ polycyclic, consistent pc presentation

$w \in G$

$w = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$ $\quad i_j \neq i_{j+1}$

- $w$ is a COLLECTED WORD if

$i_1 < i_2 < \dots < i_r$.

- A subword $v$ of $w$ is a **MINIMAL NON-NORMAL SUBWORD**

if:
- $v = x_{i_j}^{a_j} \cdot x_{i_{j+1}}^{\pm 1}$ $\quad i_j > i_{j+1}$
- $v = x_{i_j}^{a_j}$ $\quad a_j \notin \{0, 1, \ldots, r_{i_j} - 1\}$

$r_{i_j} = |G_{i_j} : G_{i_{j+1}}|$

ex $(S_4)$: $x_1, x_2, x_3, x_4$

$$x_1^2 = x_2^3 = x_3^2 = x_4^2 = 1$$

$$x_2^{x_1} = x_2^2, \quad x_3^{x_1} = x_4$$

$$x_3^{x_2} = x_1, \ldots, x_4^{x_1} = x_3$$

$$x_1^{x_2} = x_3 x_4$$

$$x_2 x_1 = x_1 x_2^2$$

Non collected word

$$x_2^{x_1} = x_2^2$$

$$x_1^{-1} x_2 x_1 = x_2^2$$

$$x_4 x_3 x_2 x_1 = x_1 (x_4 x_3 x_2)^{x_1}$$
$$= x_1 x_4^{x_1} x_3^{x_1} x_2^{x_1}$$
$$= x_1 x_2 (x_3 x_4)^{x_2^2}$$

$x_3, x_4$ commute. collect

$$= x_1 x_2^2 \, x_3 x_4 \, x_4 x_3 \, x_4$$
$$= x_1 x_2^2 \, x_4$$

GAP Demo:

pcgs := Pcgs(g)

$x_1$ = pcgs[1]; ... ; $x_4$ = pcgs[4]

x4 * $x_3$ * $x_2$ * $x_1$

> f1 * f2^2 * f4 ← GAP collected it.

```
# Collection;

pcgs:=Pcgs(g);
x1:=pcgs[1];
x2:=pcgs[2];
x3:=pcgs[3]
x4:=pcgs[4];
x2*x1;
x2^-1*x4*x1;
x4*x3*x2*x1;
```

## Collection Strategies

Special case (but works in general):

G is a finite p-group.

pcgs : $[x_1, x_2, \dots, x_n]$

$R(x) = [p_1 \dots, p_n]$

Rels: $x_i^p = x_{i+1}^* \dots$

$i > j$ : $[x_i, x_j] = x_{j+1}^* \dots, x_n^*$

$x_i x_j = x_j x_i$ RHS ✱

Minimal non-collected words:

$u = x_j^p$

$u = x_i x_j \quad , i > j.$ ✱

Q: What if you have more minimal non-collected, how do you decide which one to collect first?

1) COLLECTION TO THE LEFT (Hall, 1934)

First move all $x_1$ to the left, then $x_2$...

2) COLLECTION FROM THE RIGHT (Harvois, Nicholson 1976)

First collect the minimal non-collected subword closest to the end of the word.

3) COLLECTION FROM THE LEFT

First ... closest to the beginning of the word.

most efficient one usually

("FromTheLeftCollector" is what we did in prev. lab)

$$ex(D_{16}) = pc \langle x_1, x_2, x_3, x_4 \rangle$$

$$x_1^2 = 1$$
$$x_2^2 = x_3$$
$$x_3^2 = x_4$$
$$x_4^2 = 1$$
$$[x_2, x_1] = x_3 x_4$$
$$[x_3, x_1] = x_4 \rangle$$

$$x_3 x_1 = x_1 x_3 x_4$$

TO THE LEFT

$$
\begin{aligned}
x_3 x_2 x_1 &= \underline{x_3 x_1} x_2 x_3 x_4 \\
&= x_1 x_3 x_4 \underline{x_2} x_3 x_4 \\
&= x_1 x_2 x_3^2 x_4^2 \\
&= x_1 x_2 x_4 .
\end{aligned}
$$

FROM THE RIGHT

$$
\begin{aligned}
x_3 x_2 x_1 &= x_3 \underline{x_2 x_1} \\
&= \underline{x_3 x_1} x_2 x_3 x_4 \\
&= x_1 x_3 \underline{x_4 x_2} x_3 x_4 \\
&= x_1 \underline{x_3 x_2} x_4 x_3 x_4 \\
&= x_1 x_2 x_3 \underline{x_4 x_3} x_4
\end{aligned}
$$

FROM THE LEFT

$$
\begin{aligned}
\underline{x_3 x_2} x_1 &= x_2 \underline{x_3 x_1} \\
&= \underline{x_2 x_1} x_3 x_4 \\
&= x_1 x_2 x_3 \underline{x_4 x_3} x_4 \\
&= x_1 x_2 x_3^2 x_4^2 \\
&= x_1 x_2 x_4
\end{aligned}
$$