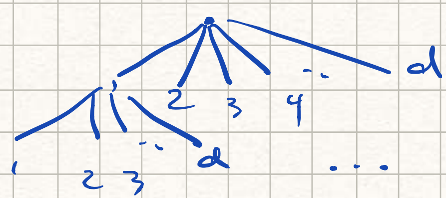
Self-Similar Groups

- Can be described as action on a tree.
- Finite L-presentation.

Automorphisms of a d-ary tree

G, $\forall g \in G$ $\exists n$ st $g^n = 1$.

Consider an infinite reg. rooted tree.

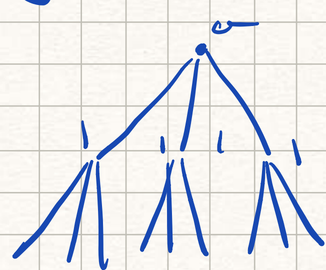


$\mathrm{Aut}\, T_d \simeq \mathrm{Aut}\, T \sim S_d$

Let $g = (1, 1, 1)\, \sigma = \sigma$

the rooted automorphism corresponding to σ.

this tree is the MAP (that's why bunch of 1's)



• $f \in \mathrm{Aut}\,T$

$f = (f_1, \dots, f_n)\sigma$

f_i section of f at the vertex i

$f = (f_1, f_2, \dots, f_d)$

$g = (g_1, g_2, \dots, g_d)$, $g^1 = \sigma \to S_d$?

$f^g = (f_1^{g_1}, \dots, f_d^{g_d})$

$f^{g^1} = (f_1, \dots, f_d)^\sigma$

$= (f_{\sigma(1)}, \dots, f_{\sigma^{-1}(d)})$

when we conjugate by an element of Aut, it's like changing indices by perm.

$u \in T$.

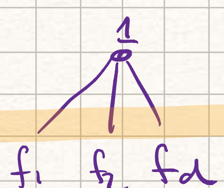
$St(u) = \{ f \in Aut(T) , f(u) = u \}$

$st(u) = \{ f \in Aut(T) ,$

$f(u) = u$

$\forall u \in L_n$

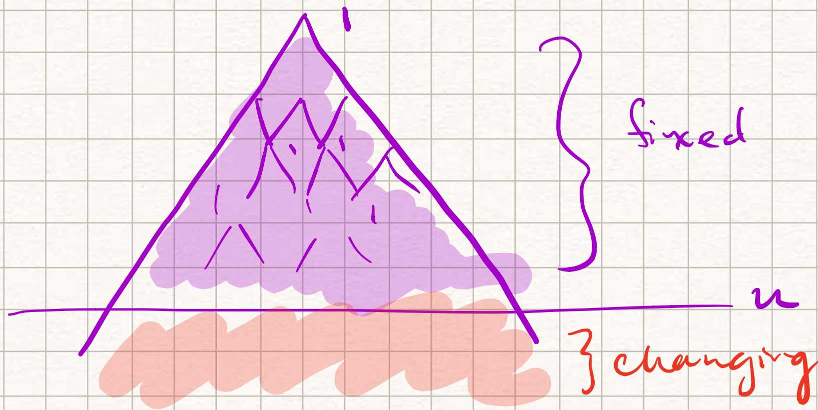
nth level of tree $\}$



$f = (f_1, f_2, f_3) \Rightarrow f \in St(1)$

because doesn't move that level.

A portrait of n'th level stabilizer:



Question:

given $f = (f_1, f_2, \dots, f_d)\sigma$
$\in$
G
is $f_i \in G$?

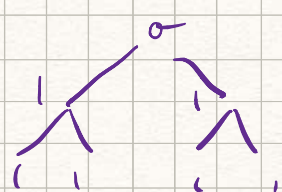
$\Rightarrow$ SELF-SIMILAR GROUPS!

$G \leq \mathrm{Aut}\, T$ is self-similar
f $\forall u \in T \quad \forall g \in G$
$\Rightarrow g_u \in G$.

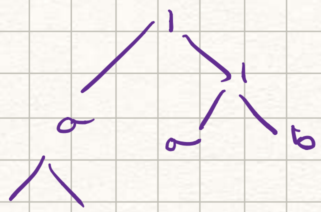
Γ = GRIGORCHUK GROUP
(R. GRIGORCHUK, 1980)

T_2 binary tree, $\Gamma = \langle a, b, c, d \rangle$

$\sigma = (0\,1)$



$b = (a, c)$
$c = (a, d)$ recursive definition
$d = (1, b)$



etc

this is not easy to write in GAP.

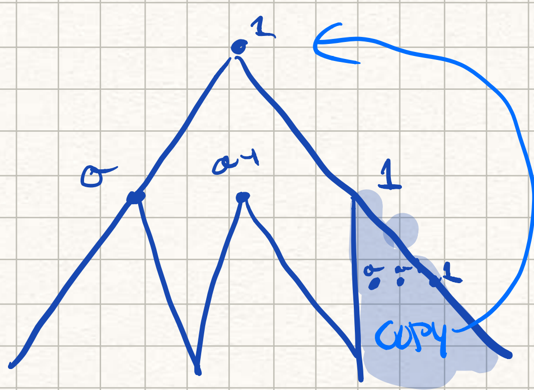
Let's see how we'd write this in program.

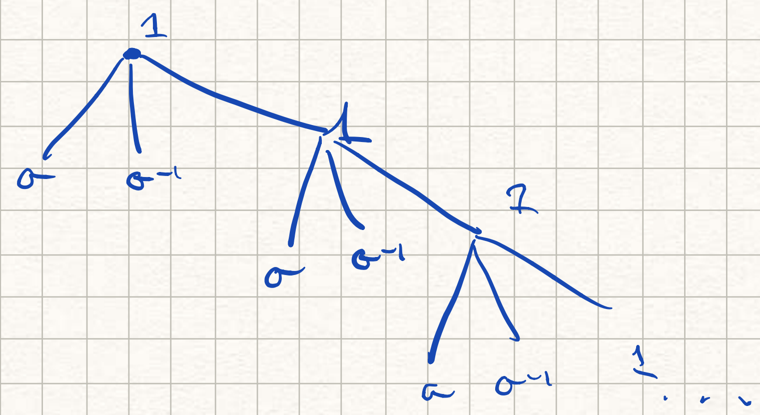
Aut T

Aut T $\quad f = (f_1, f_2, \ldots, f_d)\sigma$

$b = (a, a^{-1}, b)$

↑ self-similar





GAP demo

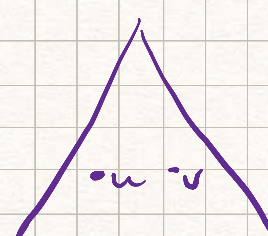
```
LoadPackage("automgrp");
gs := SelfSimilarGroup("
        a=(1,1,1)(1,2,3), b=(a,a^-1,b)");

IsFinite(gs);
> false.
```

Def: A group is spherically transitive if

$u, v \in L_n$

$\exists g \in G : g(u) = v$

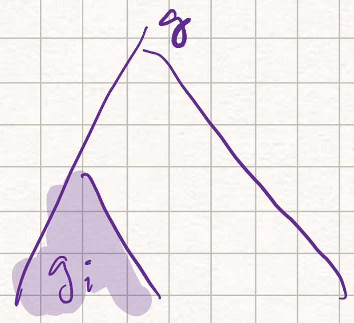


It is contracting

$\exists \lambda, k$ st $\forall g \in G$

$|g_i| \leq \lambda |g| + k$.

Paper showed that this implies solv. WP.



IsContracting(gs);

> true.

Section(b, [3,3,3]);

> b

PermOnLevel(a,2);

Decompose(b*a, 2);

StabilizerOfLevel(gs, 2);

FindElements(gs, Order, 3, 5);

at most of length 5

These groups are finitely generated,

but NOT finitely presented.

L-presentation:

$L = \langle S \mid Q \mid \Phi \mid R \rangle$

S alphabet

Q, R set of reduced words $\in F_S$.

$\Phi = \{\varphi : F_S \to F_S,\ \varphi \text{ endomorphism}\}$.

A group G admits a presentation if $G = F_S / \langle Q \cup \underset{\varphi \in \Phi^*}{\varphi(R)} \rangle^{F_S}$

Thm: If G is self-similar, contracting, regular branch, fin gen, then G admits a finite L-pres, but G is not finitely presented.

Def: G self-similar, spherically transitive regular branch if $K \le G$,

$|K : G| < \infty$

$K^X \supseteq {}^X K \le K$.

1985

$S = \{a, c, d\}$

$Q = \emptyset$

$R = \{a^2, (ad)^4, (adacac)^2\}$

$\Phi = \{\sigma\}$

$\sigma : F_S \to F_S$

$a \to aca$

$c \to cd$

$d \to c$

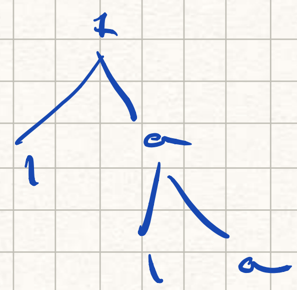
$\Gamma = \langle a, c, d \mid \sigma^i(a \ldots \quad \rangle$

Basilica group (Grigorchuk).

$B = \langle a, b \rangle$

$a = (1, b)$

$b = (1, a)\sigma$, $\sigma = (12)$



Amenability by rand. walk (2003)

$\tilde{B} = \langle a, b \mid \sigma^i([a, a^b]),\ i \geq 0 \rangle$

$\sigma : \{a, b\}^* \to \{a, b\}^*$

$a \mapsto b^2$

$b \mapsto a$