Polycyclic Groups (PART 5)

Recall : GAP exercise.

$G = \langle a,b \mid [x^4,y] = 1 \quad \text{nilpotent} \rangle$

$\Rightarrow G/Z(G)$ exp dividing 4.

means it's locally finite.

finite 2-group

(look at "Burnside problem")

proved

$[a,b]^8 = 1.$

ie, any grp w/ this prop is homomorphic image of G

freest one.

Thm exp $G/Z(G) \mid 4 \Rightarrow$ law $[x,y]^8 = 1.$

Q: Can we say something about exponent of derived subgroup?

GAP demo :

```
G := NilpotentQuotient(f/[Comm[x^4,y]],
        idgens := [x,y]);

Order(Comm(a,b));
>8

(a*b)^16
> a^16 * b^16
```

We can deduce

$(ab)^{16} = a^{16} b^{16}$

$\Rightarrow \exp G' \mid 16.$

GAP can be useful to prove theorems!

ex: $a \in G$ is a right n-Engel element if:

$$[a, \underbrace{x, \ldots, x}_{n}] = 1 \quad , \quad \forall x \in G.$$

<a, b | a & b have right 3-Eng nilpotency>

(demo):

G:=NilpotentQuotient
(f/ $\underbrace{\text{LeftNormComm}}_{\text{custom fn}}$[b, x, x, x] :
id gens := [x])

NilpotencyClassOfGroup(G);

> 4

PrintPcpPresentation(G);

deduce:

can prove once you know grp is largest.

We know from GAP

Pcp of ord $[0, 0 \ldots, 0, 2]$

and $\underline{g_6} = [g_4, g_2]$

$= [g_3, g_1, g_2]$

$= \underline{[b, a, a, b]}$

this used to be a paper-worthy theorem!

Can also prove things for infinite grps.

G is an fp-group.

G^{ab} infinite $\Rightarrow$ G infinite

Thm (Newman)

Let G be a fp group.

Let G be elementary abelian of rank d,

$P_1(G)/P_2(G)$ abelian of rank p.

Let G be presented with b generators and r relations.

If any of the following holds

i) $r - b \leq \frac{d^2}{4} - d$

ii) $r - b < \frac{d^2}{2} + (-1)^p \frac{d}{2} - d - e$

iii) $r - b < \frac{d^2}{2} + (-1)^p \frac{d}{2} - d - e$
$+ \left(e - (-1)^p \frac{d}{2} - \frac{d^2}{4}\right) \frac{d}{2}$

Newman Infinity criterion

then G has no largest p-quotient.

Then G is infinite.

demo:

```
g := FibonacciGroup(2,9);
AbelianInvariants(g);
> [2,2,19]
```

oh no! It's finite!

Try something else

we know:

$G/\gamma_3(G)$ finite

↑ will try to prove γ_3 is infinite.

$\gamma_3(G)$ =

```
k = Kernel(epi);
NewmanInfinityCriterion(k, 5)
> true;
```

$5 = p$

often will return Error trivial quotient if we pick wrong grp or p.

p-Group Generation Algorithm

'construct large p-groups.

Let G be finite p-group of p-class c, d generated.

<u>Def</u>: H is a DESCENDANT of G if

$$d(H) = d$$
$$H/P_c(H) \cong G$$

H is an IMMEDIATE DESCENDANT of G if it's a descendant of p-class $c+1$.

<u>Algorithm</u>: given G, compute immediate of G.

! careful. may give redundant descendants

<u>easy part</u>: compute

<u>hard part</u>: not get redundancy

we will only do this.

$G = F/R$ (consistent pc presentation)

1) Compute the p-cover of G:

$$G^* = F/R^* \, , \quad R^* = [R, F] R^p .$$

making things nice w/ adding gens & rels (see prev lectures)

Thm: Every immediate desc of G is an image of G^*.

Def: $\underbrace{R/R^*}_{\text{generator independant!}}$ is an elementary abelian p-group, the

p-multiplier of G

$P_c(G^*) \leq R/R^*$ nucleus of G; $N(G)$

Def: A subgroup $U \leq R/R^*$ is allowable if G/U is an immediate descendant of G.

Thm: U is allowable $\iff$ U supplements $N(G)$ in R/R^*.

$$U \cdot N(G) = R/R^* .$$

ex (D_{16}):

2-cover of D_{16}:

$D_{16}^* = \mathrm{pc}\langle a_1, \dots, a_7 \mid$

$a_1^2 = a_6,\ a_2^2 = a_3 a_4 a_7,$
$a_3^2 = a_4 a_5,\ a_4^2 = a_5,$

$[a_2, a_1] = a_3,\ [a_3, a_1] = a_4,$

$[a_4, a_1] = a_5$ + trivial $\rangle$

2 mult

$R/R^* = \langle a_5, a_6, a_7 \rangle$ abelian of ord 8

$N = P_3(D_{16}^*) = \langle a_5 \rangle$

$\Rightarrow$ linear alg problem.

allowable subgroups

$U_1 = \langle a_6, a_7 \rangle$

$U_2 = \langle a_5 a_6, a_7 \rangle$

$U_3 = \langle a_6, a_5 a_7 \rangle$

3 immediate descendants

$D_{16}^* / U_1 = \mathrm{pc}\langle a_1, \dots, a_5 \mid$

$a_1^2 = 1,\ a_2^2 = a_3 a_4,$
$a_3^2 = a_4 a_5,\ a_4^2 = a_5$

commutator rels
+ trivial rels $\rangle$

etc.

demo

```
LoadPackage("anupq");
PqDescendants(DihedralGroup(16) :
    Prime := 2, CapableDescendants);
```

NOTE: It used to be an open problem whether

$$|G| \,\Big|\, |\mathrm{Aut}(G)|.$$
?

this is FALSE by Jaikin & Al.

but we only know existence of that grp, not the grp itself

→ Can try to find it in GAP! make big p-grps.

demo

Package loaded by default "autpgrp"

saves grp as record & easy to compute Aut of p-grp.

The END

(Join the GAP forum!)