1 - Hyperbolic Spaces

Let G be a grp $G \curvearrowright S$ metric space by isometries.

This action is called

- Proper if:
  $\forall\, r > 0,\ \forall\, s \in S$
  $$|\{g \in G \mid d_S(s, gs) \leq r\}| < \infty.$$

- Cobounded if: $\exists$ a bounded $B \subseteq S$ st
  $$S = \bigcup_{g \in G} gB$$

- Geometric if its proper & cobounded.

A map $f : (S, d_S) \to (T, d_T)$ is a quasi-isom if. $\exists\, C > 0$ st

(i) $\forall\, x, y \in S$
$$\frac{1}{C} d_S(x,y) - C \leq d_T(f(x), f(y)) \leq C \cdot d_S(x,y) + C$$

(2) $f(S)$ is C-dense in T.

ex: G is fg, X, Y are finite gen sets, then $(G, d_X) \underset{qi}{\sim} (G, d_Y)$.

ex: If G is fg, $H \underset{fi}{\leq} G$ re-listen.
$\Rightarrow H$ is fg and $G \underset{qi}{\sim} H$.

Note: $H \curvearrowright \mathrm{Cay}(G)$ geometrically.

Thm (Fremovich) Schwartz-Milnor lemma
Suppose that a grp G acts co-boundedly on a geodesic m.s S.
Then, $\exists$ a generating set $X \subseteq G$ st the Cayley graph

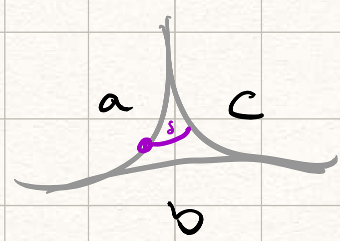
$$\Gamma(G, X) \underset{qi}{\sim} S.$$ review this pf.

If in addition, $G \curvearrowright S$ is proper, then $|X| < \infty$.

ex: If M is a compact manifold, then $\pi_1(M) \sim_{qi} \tilde{M}$.

$M = S^1 \times S^1$ $\qquad\qquad$ $\tilde{M} = \mathbb{R}^2$

Def: A m.s S is called hyperbolic if
(1) S is geodesic
(2) $\exists\, \delta > 0$ st $\forall$ geod $\triangle$ w/ sides a, b, c $\forall x \in a$, $\exists y \in b \cup c$ st

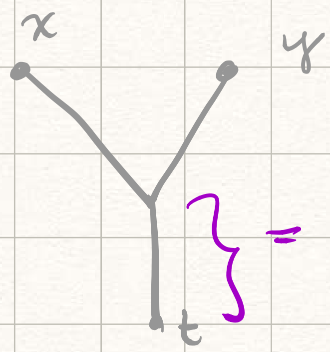


$d_S(x, y) \leq \delta$.

ex: (1) Bounded spaces ($\delta = \operatorname{diam}(M)$)
(2) $\mathbb{H}^n$
(3) Trees ($\delta = 0$).
(4) $\mathbb{R}^n$ is NOT hyp. for $n \geq 2$.

Def: Given $x, y, t \in S$, the Gromov product of x, y w.r.t t is:

$$(x, y)_t := \frac{1}{2}\left(d(x, t) + d(y, t) - d(x, y)\right)$$

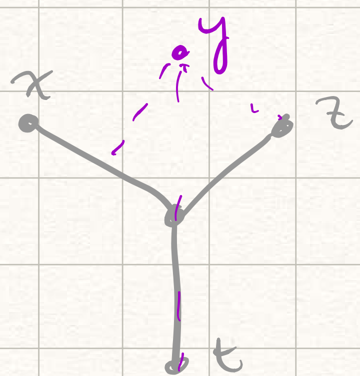


$= (x,y)_t$ len of common part.

Hw #1: If S is a tree, then $\forall x,y,z,t \in S$, we have

$$(x,z)_t \geq \min\{(x,y)_t, (y,z)_t\}$$

hint:



Thm: A ms S is hyperbolic $\iff \exists \delta > 0$ st $\forall x,y,z,t \in S$, we have

$$(x,z)_t \geq \min\{(x,y)_t, (y,z)_t\} - \delta.$$

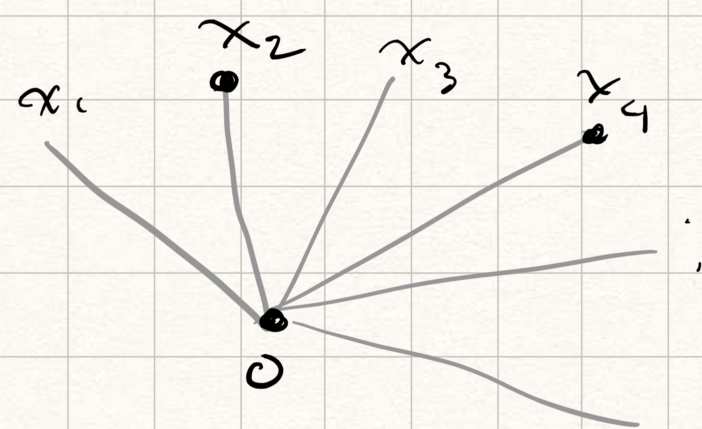
Hw #2: Prove this. (Also well-known fact).

1.1 - Gromov boundary

Def: Let S be a hyp. space. Fix $o \in S$.
A sequence $(x_i) \in S$ converges to ∞ if $\liminf\limits_{i,j \to \infty} (x_i, x_j)_o = \infty$

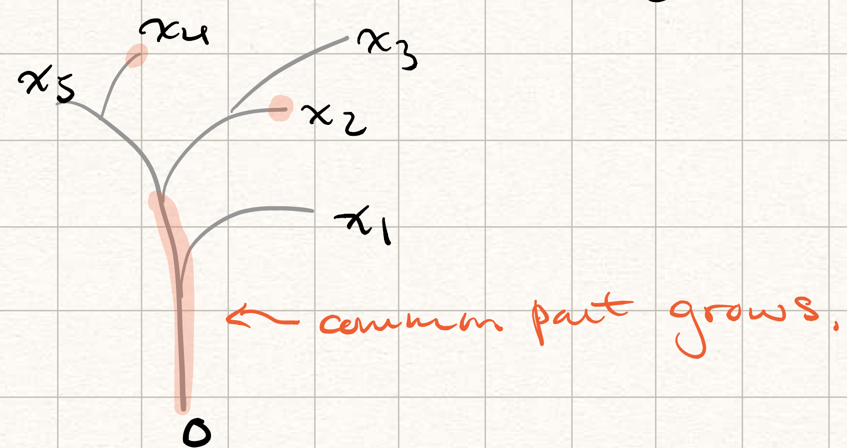
← Gromov prod.

ex:



$(x_i, x_j)_o = 0$ so $(x_i, x_j)_o \not\to \infty$.

ex:



Def: We say that $(x_i) \sim (y_i)$
if $\liminf\limits_{i,j \to \infty} (x_i, y_j)_o = \infty$

if they have a growing common part.

HW #3 : $\sim$ is an equiv. rel. on the set of sequences $\to \infty$.

Def: $\partial_0 S = \{ [(x_i)] \mid (x_i) \to \infty \}$

Add comment.

Now define Gromov prod on $\partial_0 S$ st larger prod $\Rightarrow$ closer pts.

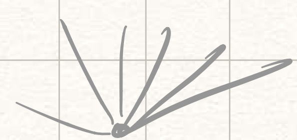
Def: The base of nbhds of $a \in \partial_0 S$ is $U_a(r) = \{ b \in \partial S \mid \exists (x_i) \in a, (y_i) \in b$ st $\liminf_{i,j \to \infty} (x_i, y_j) \geq r \}$

ex: (1) ∂_0(bounded space) $= \emptyset$

(2) $\partial_0 \mathbb{H}^n \cong S^{n-1}$

(3) T locally finite tree

$\partial_0(T) =$ Cantor set.

(4) $\partial_0 \Big($  $\Big)$

↑ countably many rays

$=$ discrete (∞) set.

Note: Not compact.

(5) (Problem Session)

$\partial_o T_\infty = ?$, where T_∞ is a tree w/ every vertex of deg ω.

$\underline{Q}$: Can you use geod. rep for $\partial_\delta S$?

$\underline{A}$: No. Not sufficient.

(Leave as exercise for tmrw).

Thm (Gromov) If S is hyperbolic, then (a) $\partial_o S$ is independent of the choice of bpt o up to homeo.

(b) $\underline{\partial} S$ is a completely metrizable space.

no subscript

If S is proper, then ∂S is compact.

(c) If R is geodesic and $R \sim_{qi} S$, then $\partial R \cong \partial S$.

(d) If $G \curvearrowright S$, then this action induces $G \curvearrowright \partial S$ by the rule

$g : g[(x_i)] \longmapsto [g(x_i)]$.

1.2 - Grp actions on hyp spaces

Def: An elmt $g \in G \curvearrowright S$ is

- elliptic if $\langle g \rangle$ has bounded orbits
- parabolic if $\langle g \rangle$ has unbounded orbits and fixes 1 pt on ∂S.
- loxodromic if $\langle g \rangle$ has unbounded orbits and fixes exactly 2 pts in ∂S.

Thm (Gromov) Every isom. of a hyp space is elliptic, parabolic or loxodromic.

ex: $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$,

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} z = \frac{az+b}{cz+d}$$

- If $|\text{tr } A| < 2$ or $A = \pm I$, then A is elliptic.
- If $|\text{tr } A| = 2$, then A is parabolic.

- If $|\mathrm{tr}\, A| > 2$, then A is loxo.

Def: Let $G \curvearrowright S_{hyp}$. Given $R \subseteq S$, define $\partial R \subseteq \partial S$,

$$\partial R = \{ [(x_i)] \in \partial S \mid (x_i) \subseteq R \}$$

Def: Fix $o \in S$. Let $G_o = \{ go \mid g \in G \}$. ($\partial$ orbit of $o \in S$)

The limit set of G is $\Lambda(G) = \partial G_o$.

Rem: $\Lambda(G)$ is indep of o, because moving pt = moving by bounded dist, which does NOT affect ∂G_o.

Thm: $\forall G \curvearrowright S_{hyp}$, $|\Lambda(G)| \in \{0, \underset{\text{ellip}}{1}, \underset{\text{lox}}{2}, \infty\}$. ex: $F_2 \curvearrowright Cay$

Def: The action $G \curvearrowright S_{hyp}$ is:
- elliptic if $\Lambda(G) = \emptyset$ (equiv: orbits are bounded).
- parabolic if $|\Lambda(G)| = 1$
- lineal if $|\Lambda(G)| = 2$,
- quasi-parabolic if $|\Lambda(G)| = \infty$, and G fixes a pt on ∂S.
- of general type if $|\Lambda(G)| = \infty$, and G does not fix any pt in ∂S.

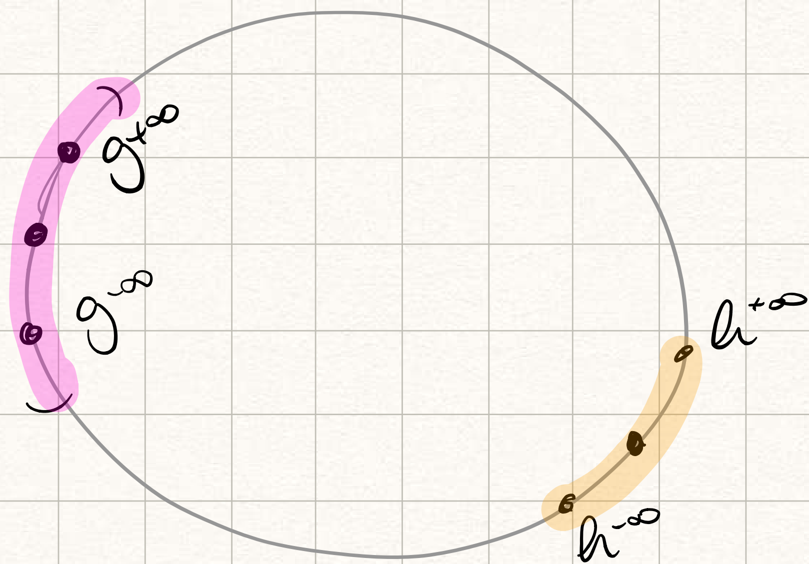
Rem: $g \in G$. G is elliptic (resp. par, lox) if $\langle g \rangle$ is ell (resp. par, lineal)

Thm: ▶ PLAY!

Action	$\lvert \Lambda(G) \rvert$	$\mathrm{Fix}_G(\partial S)$	Orbits	Types of elmts
elliptic	0	any	bounded	elliptic
parabolic	1	1	unbounded, never q-convex. quasi-conv	elliptic or parabolic.
lineal	2	0 or 2 $D_\infty \curvearrowright \mathbb{R}$	$\sim_{qi} \mathbb{R}$	elliptic or loxo.
quasi-parabolic	∞	1 HW#4. $\Longrightarrow$	unbounded q-convex	any (A) $\{g^{\pm\infty} \mid g \in \mathcal{L}(G)\}$ is dense in $\Lambda(G)$.
general type	∞	0	unbounded	any $\{(g^{+\infty}, g^{-\infty}) \mid g \in \mathcal{L}(G)\}$ is dense in $\Lambda(G) \times \Lambda(G)$. (A)

Def: Fix an action $G \curvearrowright S$.
Given a lox $g \in G$, let $\{g^{+\infty}, g^{-\infty}\} = \Lambda(\langle g \rangle)$
$\mathcal{L}(G) = \{$all lox of $G\}$

$g, h \in \mathcal{L}(G)$ are independent if
$\{g^{\pm\infty}\} \cap \{h^{\pm\infty}\} = \emptyset$



☆ $\Rightarrow$ $G \geq$ free subsemigrp of rk 2.
☆ $\Rightarrow$ $G \geq F_2$.

<u>ex</u>. $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$.
(1) $SO_2(\mathbb{R}) \cong \mathbb{R}/\mathbb{Z}$ is elliptic.

(2) $\mathbb{Z} \cong \{\underbrace{\begin{pmatrix} e^n & 0 \\ 0 & e^{-n} \end{pmatrix}}_{=A} \mid n \in \mathbb{Z}\}$
$|tr(A)| = e^n + \frac{1}{e^n} \geq 2$.
A is lineal

(3) $UT_2(\mathbb{R})$ → triang matrix w/ diag entries 1.

$UT_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ is parabolic.

(4) $ST_2(\mathbb{R})$
$= SL_2(\mathbb{R}) \cap T_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ is quasi-parab.

(5) $SL_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ is of general type.

ex: $G \curvearrowright$ tree. Then every elmt is either elliptic or lox. (More gen, fg grps cannot act parabolically on a tree).

ex: $G = A *_C B \curvearrowright$ Bass-Serre tree. The action is
- elliptic $\Leftrightarrow$ $A=C$ OR $B=C$.
- lineal if $|A:C|=2$, $|B:C|=2$.
- general type otw.

Hw # 5: $BS(1,2) = \langle a,b \mid b^{-1}ab = a^2 \rangle$
is HNN-ext. of grp $\mathbb{Z}$

What type does $BS(1,2)$ on the B-S tree have?

Next time: discuss hw.