Intro 3-manifs

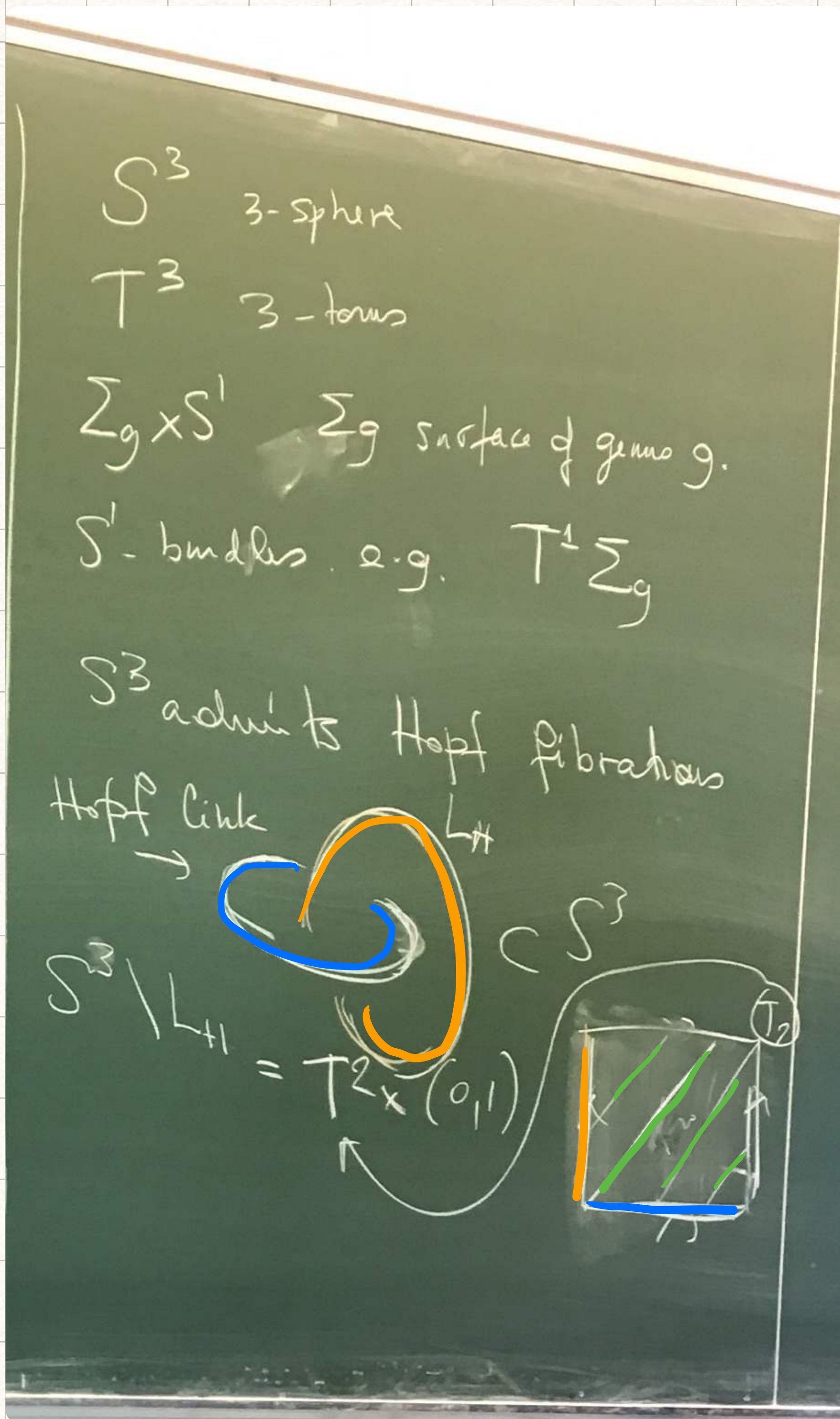
If you want to understand 3-manifs, start by understanding free surface grps and their automorphisms.

→ 2 fams of ex.
- Seifert manifs
- mapping tori

→ decomp along sphere & tori
→ hype. 3-manifs and virt. props.

☆ All manifolds will be connected, oriented, mostly compact (closed or w/ boundary) or cusped (a manifold obtained by removing tori component of the ∂).

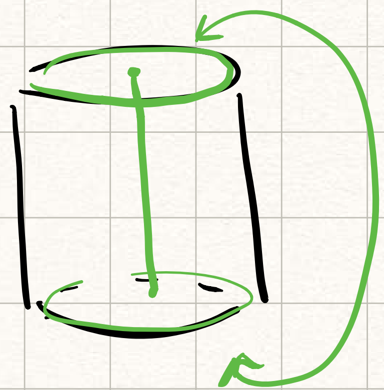
ex:



☆ Choose natural slope on T^2
$\neq$ vert & hor: this gives a
recipe to subdivide $T^2 \times (0,1)$,
then S^3 into circles.

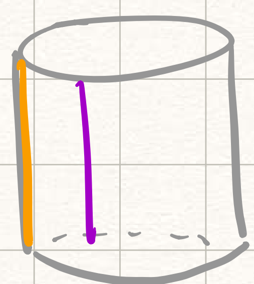
Def: A manifold M is a **Seifert space** if it can be seen as a disj. union of S^1, st every S^1 has a nbhd of the form

$T(\alpha, \beta)$ = a solid torus.



glue bottom to top via $2\pi \cdot \frac{\beta}{\alpha}$-rotation

where $\alpha, \beta \in \mathbb{N}$, coprime, $0 \leq \beta < \alpha$.

ex: 

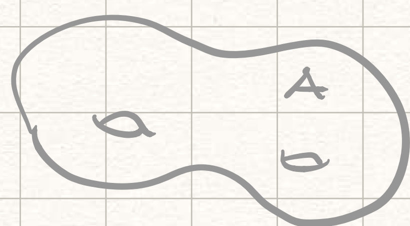
$\alpha = 2, \ \beta = 1$

This circle (= fibre) is

- **regular** if $\alpha = 1$
- **singular** of singularity of type β/α if $\alpha \neq 1$.

Rmk: This manif admit S^1 act. (translation about fibers), moreover, they can be characterized this way: The action is free along regular fibers and has cyclic stabilizers of order α along the sing. one.

→ You can consider the space of fibers / orbits of S^1-act which is 2-dim orbifolds.



Seifert manifolds are classified as manifolds / as fibered manifs:

→ either they are "small", that is, they admit several distinct fibrations, or they have

→ a unique fibration they are all geom fundamental. grps of Seifert manifolds.

→ finite grps that are subgrps of $O(4)$ acting freely on S^3.

Moreover, b/c of Poincaré's geometrization conjecture, if $\pi_1(M)$ finite, that is, M is a Seifert-fibered space as quotient of S^3.

If the grp infinite, then it is central ext of a Fushian grp

$$
\begin{array}{ccccccccc}
 & & \text{center} & & & & \text{base orbifold of fibration} & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(M) & \rightarrow & \pi_1(B) & \longrightarrow & 1 \\
 & & \uparrow & & & & \uparrow & & \\
 & & \text{grp generated by regular fiber} & & & & \text{Fushian grps + Euclidean orbifold grps.} & &
\end{array}
$$

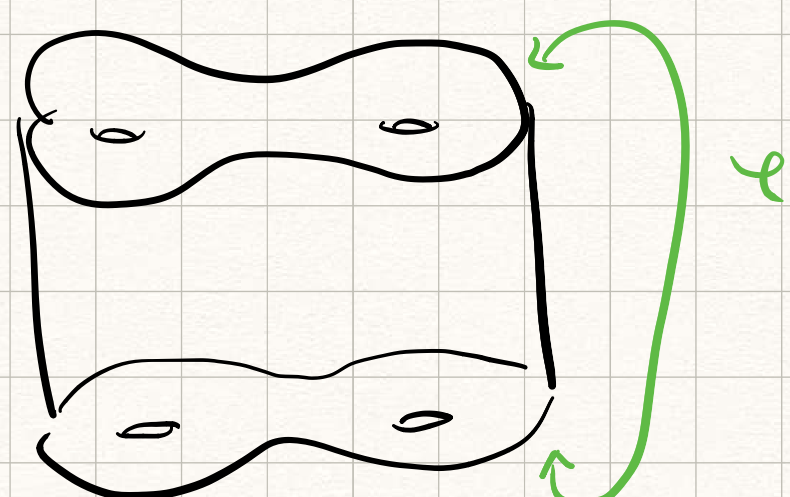
Thm: (Seifert fibered space (centre))

A compact, irred 3-manifold M w/ infinite π_1 is a Seifert manif $\Leftrightarrow$ $\pi_1(M)$ has non-trivial center.

Surface Bundles over S^1.

Def: Σ_g is a closed, connected oriented surface of genus g, $\varphi: \Sigma_g \to \Sigma_g$ a homeo.

The mapping torus M_φ is the manifold obtained from $\Sigma_g \times [0,1]$ by gluing $\Sigma_g \times \{0\}$ to $\Sigma_g \times \{1\}$ via φ.



- If $\varphi = Id_{\Sigma_g}$, then we got a product.

- A manifold can admit several structures as mapping torus.

Lemma (Gluing lemma)

Let M be a (possibly disconnected) 3-manifs, and let Σ_1, Σ_2 be two distinct but homeo components of ∂M.

Let $\varphi, \varphi' : \Sigma_1 \to \Sigma_2$ homeo.
Let $M_\varphi, M_{\varphi'}$ be the manifs obtained by id'ing Σ_1 onto Σ_2 via φ (resp. φ').

Then M_φ is homeomorphic to $M_{\varphi'}$ if one of the following conds hold:

(1) φ, φ' are conjugate via a homeo of M.

(2) φ and φ' are isotopic.

"Most of the time," mapping tori are geometric.

The fundamental grp of a mapping torus is just HNN-ext of a surface grp.

Makes sense: regluing something in itself.

Thurston's 3-dim geometries

Def: A model geometry is a Riemannian manifold X together w/ its maximal grp G of isoms.

→ We want G to be Lie grp,

→ X simply connected.

→ G acts transitively on X.

Def: A manifold M admits an
(X,G)-geometric structure
if: $M = \Gamma \backslash X$ where $\Gamma \subset G$,
Γ is discrete and acts
freely.

We are interested in
geometric models w/ compact
quotient manifs.

up to taking
finite
cover.

lower the dim.

Can find surf tang. to plane distr everywhere, integrable ie diff gen

Pt Stab		Geom	wrt to curv at every pt.	Manif up to finite orb.
SO(3)		S^3	curv +1, Seifert manif	S^3
		$\mathbb{E}^3$	curv 0, Seifert manif	T^3
		$\mathbb{H}^3$	curv −1	mapping tor of pseudo-A.
SO(2)	Int'able plane distr. why?	$S^2 \times \mathbb{R}$	Seifert fib'ed	$S^2 \times S^1$
		$\mathbb{H}^2 \times \mathbb{R}$	Seifert fib'ed	$\Sigma_g \times S^1$, $g \geq 2$
	Non int'able	Nil	Seifert fib'd $1 \to \mathbb{R}^1 \to Nil \to \mathbb{R}^2 \to 1$	mapping tor. of reducible parabolic maps of T^2 or circle bundles over T^2 w/ non-zero Euler char.
		$\widetilde{PSL}(2,\mathbb{R})$	Seifert fibered	circle bundles over Σ_g, $g \geq 2$ and non-Euler char.
SO(1)		Sol	" " $1 \to \mathbb{R}^2 \to Sol \to \mathbb{R} \to 1$ $(x,y)t = (e^t x, e^{-t} y)^{(x,y)}$	Anosov mapping tori.

start surf	how you glue	result
Surface	homeo type	mapping torus
S^2	Id Alex's thm + gluing lem.	$S^2 \times S^1$
T^2	Periodic	T^3 up to finite cover
	Reducible	Nil-manif
	Parabolic	Sol-manif
$\Sigma_g, g \geq 2$	Periodic	$\Sigma_g \times S^1$ up to finite cover.
	Reducible	NON-GEOM (admits non-trivial JSJ decomp)
	Pseudo-Anosov	hyp. manif.