Lens spaces (Finite cyclic fundamental grp.)

May have the same π_1 w/o being homotopic.

Decomposition along spheres

Def: M_1, M_2 closed, conn'd, orient. The connected sum of M_1, M_2 denoted $M_1 \# M_2$ is the oriented manif. obt'd from

$$M_1 \setminus B^3 \underset{\partial}{\cup} M_2 \setminus B^3$$ by Alex thm & gluing lem.

This is uniquely def'd.

Rmk: the op induces a commutative str. on the set of (homeo classes of) orient'd 3-manif w/ unit tang S^3

follows from Jordan-Schönflies thm.

Def: M is **prime** if it whenever $M = M_1 \# M_2$, then either M_1 or $M_2 = S^3$,
(i.e. cannot be non-triv. decomp).

Def: M (any manif) is **irred** if every embedded S^2 in M bounds a ball.

Note: • S^3 is irred but not prime.
• The only prime but not irred manif is $S^2 \times S^1$.

Rmk: W/ only two exceptions (S^3, $\mathbb{RP}^3 \# \mathbb{RP}^3$), non-prime manifolds are not geometric.

If a cover of a manifold is irred, so must be the manif.

Thm (Kneser - Milnor)

M closed, connected, orient, then:

→ either $M = S^3$ or

→ there is a finite family $M_1, \dots, M_k$ of prime manifolds st $M = M_1 \# \cdots \# M_k$, and the composition is unique up to re-ordering.

Seifert van Kampen thm

$$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k).$$

! **Thm**: A manif. $M \neq S^3$ is NOT prime $\Leftrightarrow$ $\pi_1(M)$ splits as a free product.

Pf: $\Rightarrow$ Poincaré conj.

$\Leftarrow$ Stallings.

! **Thm**: A manifold M is irred
$\iff \pi_2(M) = \{1\}$.

→ holes bounded by sphere.

π_1 = holes bounded by circ.

Pf: $\Leftarrow$ follows from Poincaré conj.

$\Rightarrow$ follows from sphere thm.

$M \neq S^3$, $\pi_1(M) = 1$, $M \# M$, $\pi_2(M \# M) = 1$,
but $M \# M$ is NOT irred.

From the grp POV, a decomp. exists if the grp has infinitely many ends.

But if you have ∞-many ends, you need not be a free prod, but it is a free amalg. prod. or HNN over finite grps.

If the grp is fp, then you have a finite decomp. (Dunwoody).
(Showed that fp grp accessible, but there are fg grps that are not accessible).

Moreover, these grps must have tors.

if the grp does NOT have tors, then one can apply a result of Grushko that says that any fg grp admits a unique decomp as a free product.

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ask someone.

A grp splits over a finite grp $\iff$ it admits an action on a tree w/o global fixed pt. and w/ finite edge stab.

Knezer-Haken finiteness prop. that can be proved using normal surfaces wrt triangulation.