There are irred manifs which are NOT geometric.

Def: A closed connect orient surface Σ embedded in M^3 is incompressible if for every embedded disk $D^2 \hookrightarrow M^3$ st $\partial D^2 \subset \Sigma$, then ∂D^2 is homotopically trivial in Σ. ⇝ No new loops.

Because of the loop thm / Dehn lemma in this context, this is equiv to requiring that the embedding of Σ is π_1-inj.

Def: Σ embedded in M is essential if Σ is not boundary parallel

↳ cannot isotop Σ to be a boundary comp

and either

$\Sigma = S^2$ not bounding a ball, or

$g(\Sigma) \geq 1$ and Σ is incompressible.

Thm: Let Σ be an essential surface in a Seifert manifold M, then Σ is isotopic to either a hor surface or a vert surface (union of fibers).

ex: $N_1 = S^3 \setminus$ figure 8-knot
$N_2 = S^3 \setminus$ trefoil knot.

$M = N_1 \underset{\partial}{\cup} N_2$ so that the longitudes go to meridians & vice-versa.

exc: prove that M is irred and non-geom..

Ingredients:

① trefoil knot is hyp.
② a is non-triv. knot ext are incompressible.
③ hyperbolic closed manifs cannot contain essential tori

④ properly embedded annuli in hyp. manifs are ∂-parallel.

Def: M 3-manif. is **atoroidal** if all $\mathbb{Z} \oplus \mathbb{Z}$ subgrp of $\pi_1(M)$ are **peripheral**, ie conjugate inside a **peripheral subgrp**

meaning of this type.

$$\Sigma \xhookrightarrow{i} \partial M, \quad i_*(\pi_1(\Sigma)) \subset \pi_1(M).$$

ie all $\mathbb{Z}^2$ happens on the boundary.

Thm (Torus theorem)

Let M be a compact, irred. Assume that there is an essential (π_1-injective, non-perif) immersion of torus in M, then either M is Seifert fibered, or M contains an essential torus.

Thm: (Toric decomp)

Let M be compact, orient, irred, either closed or w/ ∂M consisting of tori. Then there exists a finite fam $\mathcal{T}$ perhaps empty, of essential, disjoint and pairwise non-parallel tori st each component of $M \setminus \bigcup_{T \in \mathcal{T}} T$ is either atoroidal or Seifert fib'd.

incompressible

Moreover, a minimal such family is unique up to isotopy.

Note: What about Sol manif?
(More precisely, mapping tori of Anosov maps).
$\mathcal{T}$ contains a single torus.

Pt: This is how you reduce manif to geometric ones.

Rmk:

Dehn's lemma, sphere thm, torus thm, Poincaré conj all share same spirit.

$\pi_1(M) \longrightarrow M$ statement-statement

Rmk: What are the analogues of these two splittings in 2-dim?

$\rightsquigarrow$ in 2 dim, S^1 is both sphere and a torus.

$\pi_1(S^1) \neq 1$ usually, &

$\pi_1(T^2)$ hyp $\leftarrow$ very special.

The grp theoretic equiv of JSJ decomp is a decomp along virtual $\mathbb{Z}$-grps (grp w/ 2 ends).

Recent dev

Now we know how to decompose manif into geometric ones
⇝ the interesting ones (right now) are hyperbolic.

⇝ To study them, you might want to study lattices in $PSL_2(\mathbb{C})$.
What's the dif b/w $PSL_2(\mathbb{R})$?

→ Both surface grp & free grps are Gromov hyp, but that's not the case in dim 3.
→ Cusped manifs are NOT Gromov hyp, they are rel. hyp wrt peripheral grps.

Of course, grps w/ hyp closed 3-manifs are all q-isom. Maybe a more interesting way is to consider abstr. commensurability instead.
↳ don't care about bpt for manif → don't care about conj.

Def: Say G_1, G_2 two grps are **abstractly commensurable** if there are subgrps $H_1 \leq_{fi} G_1$, $H_2 \leq_{fi} G_2$ st

H_1 is isom to H_2.

Grps that are abstractly comm'able are also q-isom but the converse is NOT true.

Because of Mostow rigidity thm, the fund grp of 2 hyp closed manifs that are abstractly commensurable are "very likely" to be commensurable.

However, this implies that there are qi grps which are NOT comm'able. b/c there are arithm manifs and non-arithm manifs, and arithmeticity is commensurability invariant.

NB: In the cusped manif one, then being commensurable is equiv to being qi by a result of Schwartz.

Each commensurability class of fund grps of hyp manif, there is the fund grp of pseudo-A mapping torus.

Dynamics & hyp. manifs

Bowditch showed that a grp is Gromov hyp <=> it admits a unif convergence action on some compact, metrizable space X, that is G ↻ space of distinct triples of pts of X, prop. disc w/ cmpt quotient.

Extended to rel hyp grps are also those that have geom. finite. convergence action.

Carmon's conjecture: up to passing to fi subgrp or quotienting by normal subgrp, the only Gromov hyp. grps that admit a unif. conf. action on S^2 are the unif. lattices of $PSL_2(\mathbb{C})$.

2-dim analog was proved.

Waldhausen conj every hyp. manif is finitely covered by manif that contains an essential surface.

Assume can find immersed (q-convex) surface in manif
→ can you promote it to an embedding up to passing to some finite cover?

A: Yes if grp is LERF.