Automorphisms of free grps

Plan: Introduce basic objects and techs used in modern study of auto of free grps.

"Modern" = from 1920s on but still useful.

Want to use tops + geo methods.

Let X = space to model the fg free grp,
$F_n = F = \langle a_1 \ldots a_n \rangle$,
ie $\pi_1(X) \cong F_n$.

homotop equiv ✓

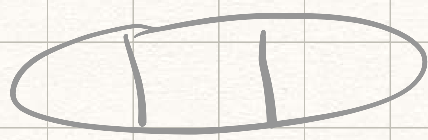
Then maps $X \xrightarrow{\sim} X$ induce

$$\begin{array}{ccc} \pi_1 X & \xrightarrow{\cong} & \pi_1 X \\ \parallel & & \\ F_n & \xrightarrow{\cong} & F_n \end{array}$$

isom.

Candidates for X :

1. X = finite graph, connected, w/ $\chi(\Gamma) = v - e = 1-n$.

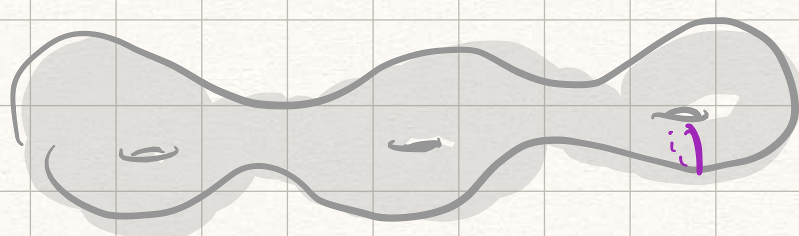


$\pi_1(X) \cong F_n$

2. Punctured surface

$\pi_1 X \cong F_{2g+s-1}$

3. X handlebody

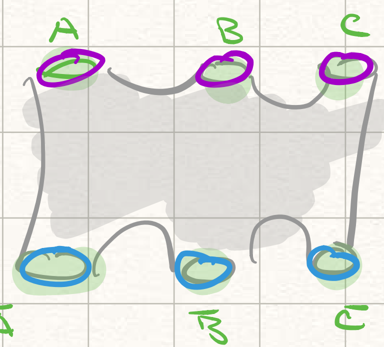


$= \#_n S^1 \times D^2$, $\pi_1 X \cong F_n$

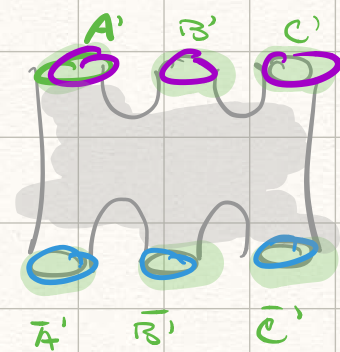
4. Double handlebody $\#_n S^1 \times S^2$

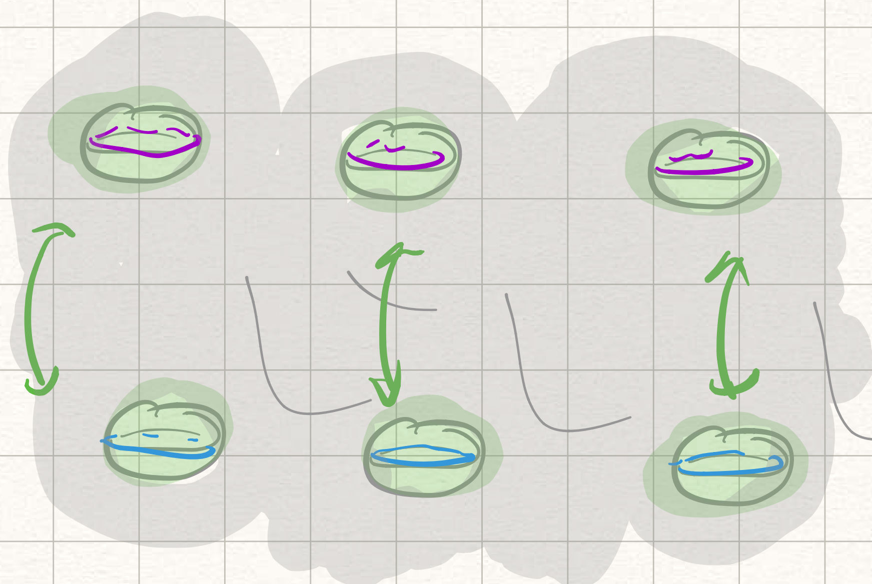
Another way to visualize

1) Open this up.



2) glue this together on green





3) these were pre-assembled together.

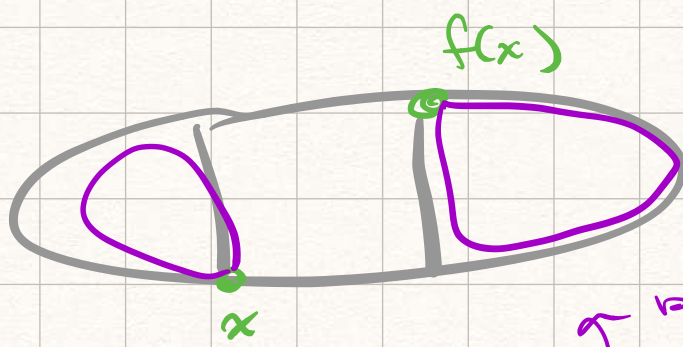
$$= S^3 - \bigcup_{2n} B^3$$

Homotop. equivalences

① $\pi_0(HE(X,b)) \to Aut(\pi_1(b))$

$\pi_0(HE(X)) \to Out(\pi_1(X))$

exc: is an isom.



② both are there

$\pi_0(Homeo(X)) \to Out(\pi_1(X))$

not surj.

③ $\pi_0(Homeo(X)) \twoheadrightarrow Out(\pi_1 X)$

not injective, ∞-kernel.

Thm: (Laudenbach)

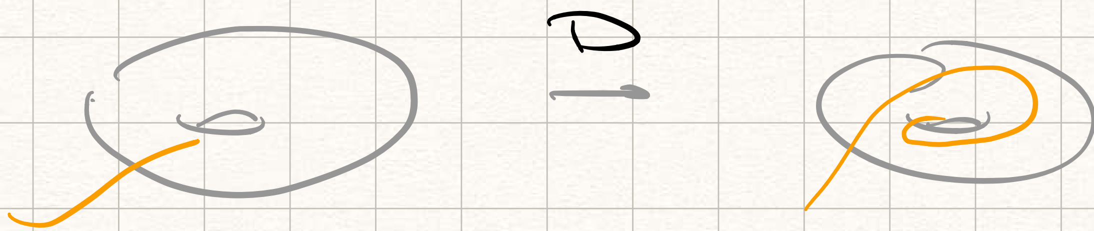
$1 \to K \to \pi_0(Homeo(M_n)) \twoheadrightarrow Out(\pi_1 M_n)$

every elmt order 2.

where K = finite 2-grp, generated by Dehn twist in 2-spher

$\cong$

$(\mathbb{Z}_2)^n$



generalize

↓

a suspension

the support of fn is entire suspension

D gives a path in $SO(3)$,

$\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$

$D^2 \longleftrightarrow \text{id in } SO(3)$

Automorphisms: (exs)

① $\iota_i : a_i \mapsto a_i^{-1}$

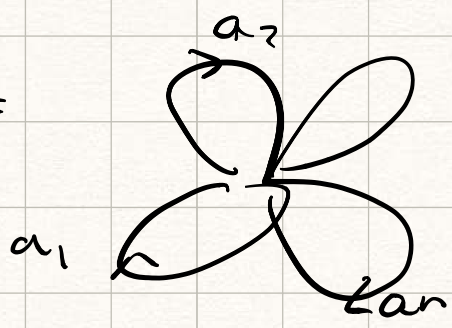
② $\sigma : a_i \mapsto a_{\sigma(i)}$, $\sigma \in Sym(n)$

③ $P_{ij} : a_i \mapsto a_i a_j$ $(a_j \mapsto a_j$ if $j \neq i)$

$\lambda_{ij} : a_i \mapsto a_j^{-1} a_i$

Graph model

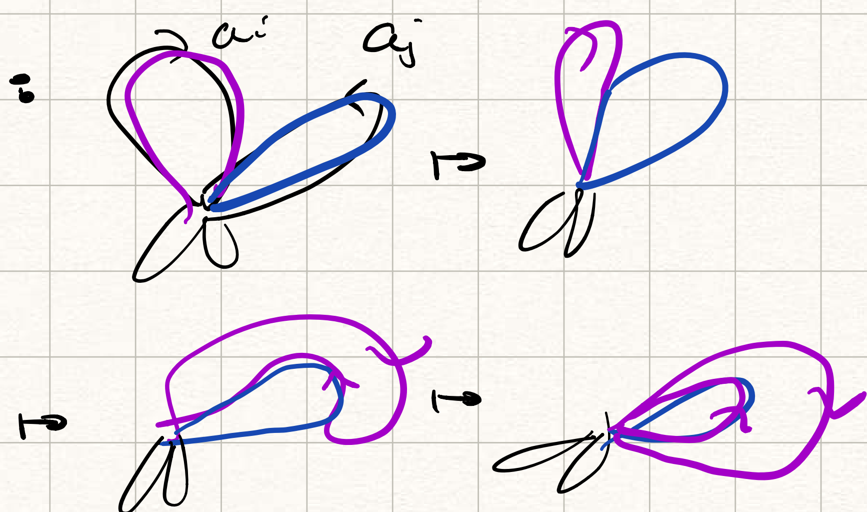
$X =$

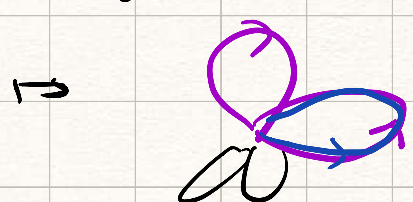


ι_i :

σ : permute the petals

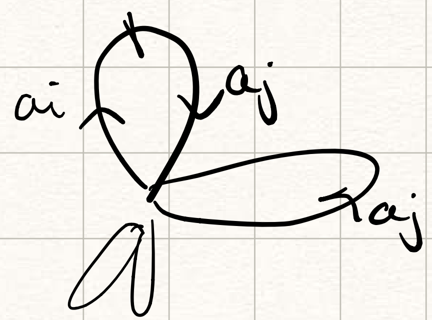
P_{ij} :





$a_i \mapsto a_i a_j$

or in one pic:



but implicit in this pic, petals are a_i's & the arrows are the maps which sends them to, say, $a_i a_j$.

This is an example of Stallings fold. (1960, "the top of finite graphs).

Thm: The auts $\iota_i, \sigma, \rho_{ij}, \lambda_{ij}$ generate $Aut(F_n)$!

Sketch: X, Y graphs w/

vertices $V(X), V(Y)$

edges $E(X), E(Y)$

$\iota \quad E(X) \to V(X)$

doubled edges $\to$ involution $E(X) \to E(X)$

$x \mapsto \bar{x}$

Def: A graph morphism $f: X \to Y$ takes $V(X) \to V(Y)$

$E(X) \to E(Y)$

commutes w/ ι & $x \mapsto \bar{x}$.

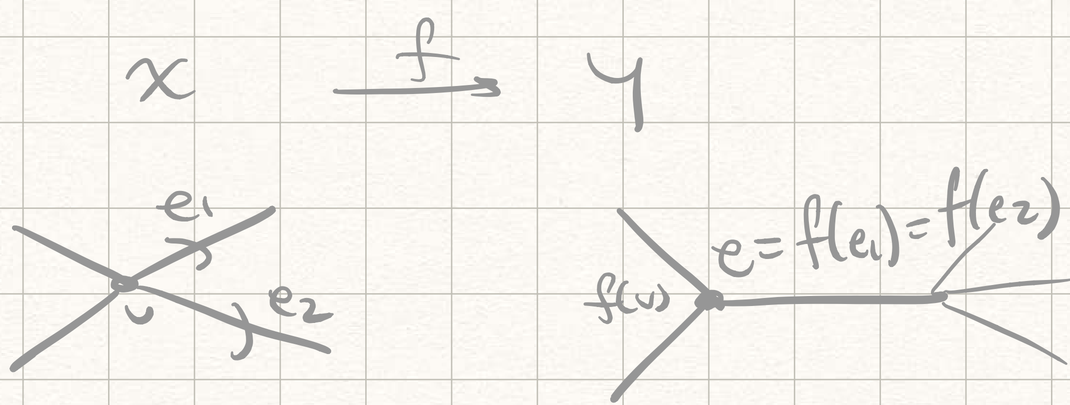
ι: injection, $x \mapsto \bar{x}$: involution.

Lemma: $f: X \to Y$ graph morphism m, if f is not locally inj, then there are

w/ $\iota(e_1) = \iota(e_2)$ and

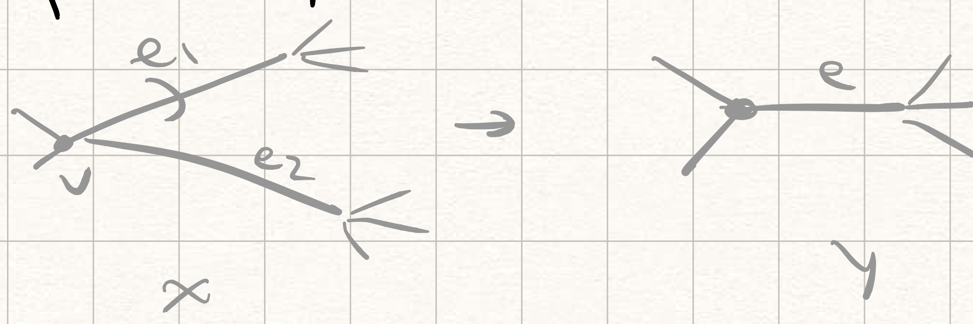
$f(e_1) = f(e_2)$

but $e_1 \neq e_2$



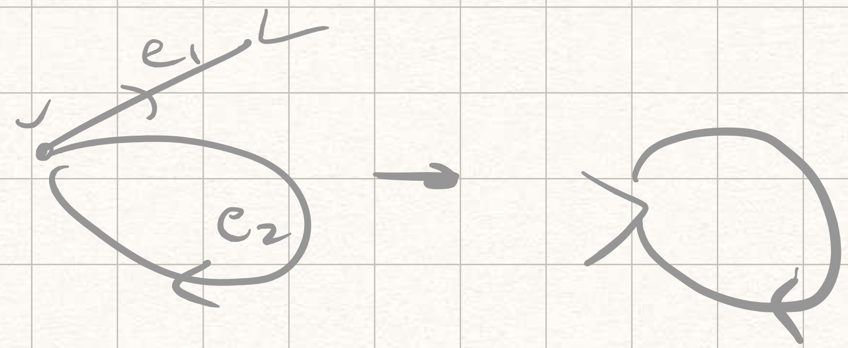
Stalling's fold: If $\iota(e_1) = \iota(e_2)$ and $e_2 \neq e_1$, then the graph morphism

1



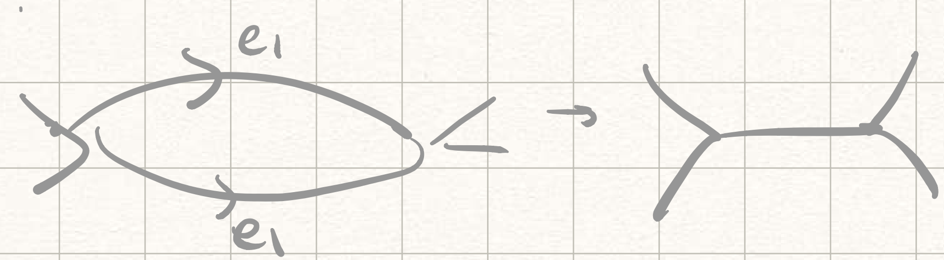
is called a stalling's fold.

ex: what we did w/ $a_i \mapsto a_i a_j$

② 

Stalling's fold called degenerate if it is not a homo. eg:

ie:

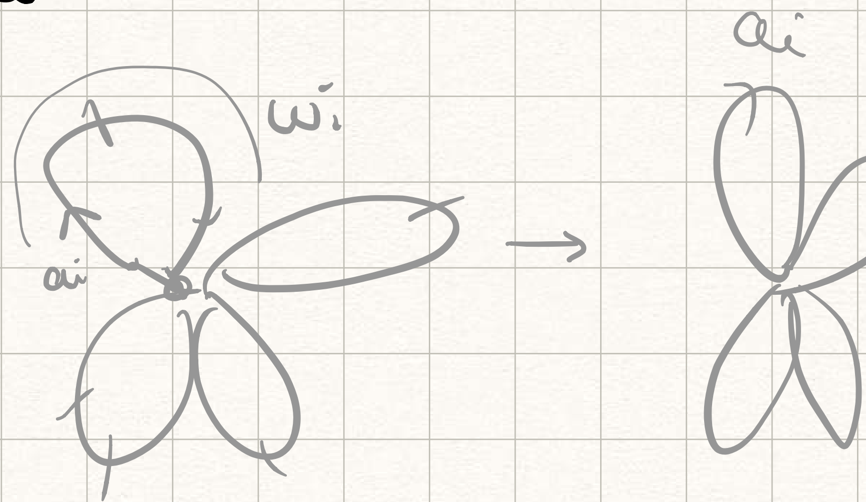


killed a loop.

Note: The exs ①, ② are only exs of NON-degenerate Stallings fold.

Let $\varphi \in \mathrm{Aut}\, F_n$

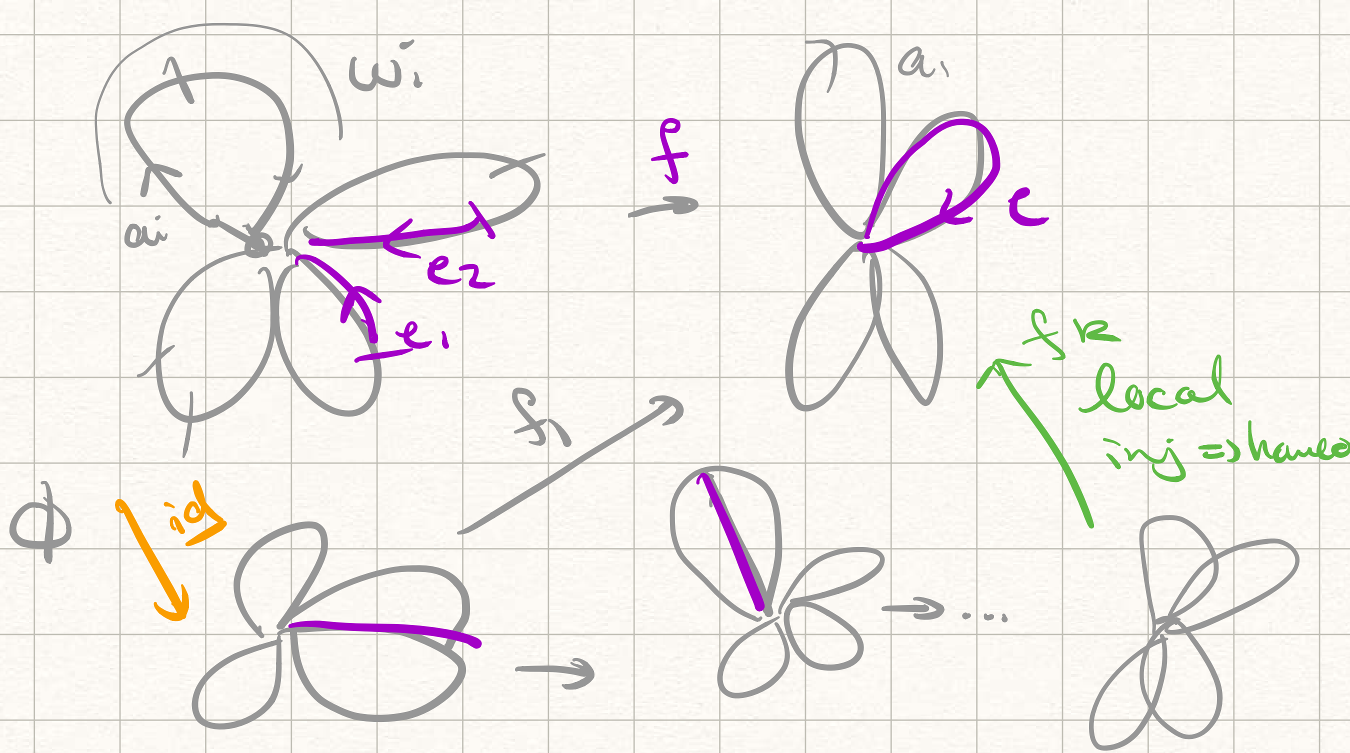
$\varphi(a_i) = w_i$



<u>Lemma</u>: f loc. inj. $\Rightarrow$ f homeomorph,

$\Rightarrow$ φ permutation + inversions.

If f is NOT loc. inj. you can fold.



A fold  induces id on π,

induces a pood of ρ_{ij}, λ_{ij}, ι