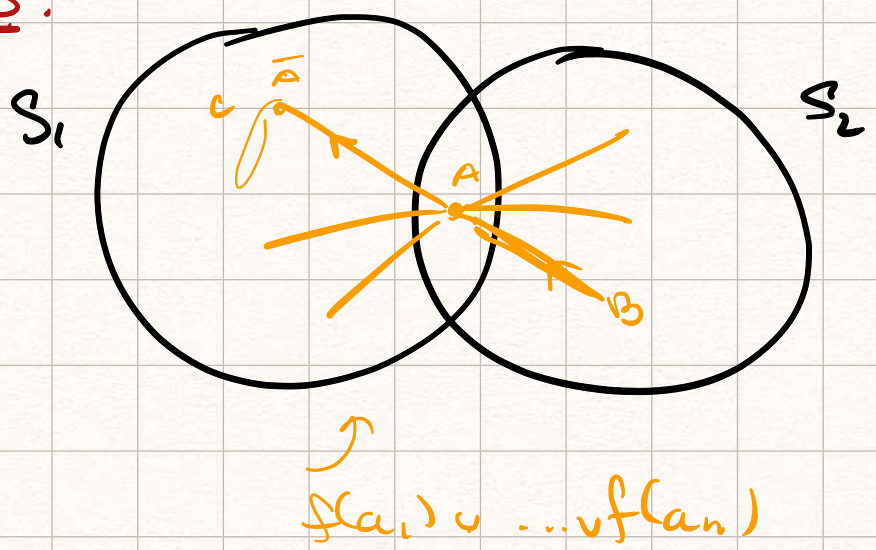
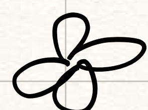
N.B:



Use S_2 sep. A from $\bar{A}$, i.e,
switch $S_2 \longleftrightarrow A$

$F_n = \pi_1(R_n =$  $)$
$\quad = \pi_1(M_n = \# S^1 \times S^2)$

We know when $f: F_n \to F_n$ is an automorphism (Whitehead).

If it is, we can factor it into a product of elmtary auts.
$\sigma_i, \iota_i, P_{ij}, \lambda_{ij}$.

<u>Now</u>: Want to understand

$$\begin{aligned} &\mathrm{Aut}(F_n) \\ &\quad \| \\ &\pi_0(HE(R_n, \theta)) \\ &\quad \| \\ &\pi_0(\mathrm{Homeo}(M_n, \theta)) / \text{Dehn twist} \end{aligned}$$

$$\begin{aligned} &\mathrm{Out}(F_n) \\ &\quad \| \\ &\pi_0(HE(R_n)) \\ &\quad \| \\ &\pi_0(\mathrm{Homeo}\, M_n) / \text{Dehn twist} \end{aligned}$$

mod out by inner Aut, which makes thus bpt indep.

Want a space X w/ actions of $\mathrm{Out}(F_n)$.

<u>GGT</u>. proper & co-compact

<u>Alg. top</u>: a <u>contractible</u> space X w/ a proper action.

We'll get $X =$ Outer space $\mathcal{O}_n$ making alg. tops happy.

But $\mathcal{O}_n$ has closely rel'd spaces,

K_n = spine of outer space

$\hat{\mathcal{O}}_n$ = bordification of $\mathcal{O}_n$

→ making alg top + ggt happy.

G is free $\Leftrightarrow$ acts freely on a simplicial tree.

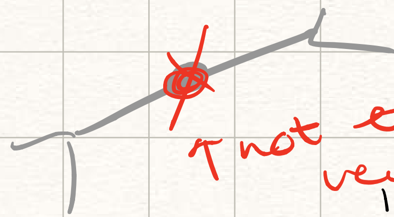
Tree = 1-dim CW complex, contractible

Actions: are graph morphisms: $E \to E$, $V \to V$

free action: every grp elmt moves every vertex.

$T = \widetilde{G}$

- only use <u>minimal</u> actions:
  no invariant subtrees
  (no bivalent vertices)



↑ not this vertex.

To make a space, put metric on T, invariant under tree action.

Use <u>equivariant Gromov-Haus top.</u>

X $\quad$ X' $\quad$ metric space

X is <u>close</u> to X' if for every finite $S \subset X$, $\exists S' \subset X'$ and $S \leftrightarrow S'$ (bij. correspondance) st $d(s_1, s_2) \sim d(s'_1, s'_2)$ w/ ε.

<u>equivariant</u>: $X \circlearrowleft G$, $X' \circlearrowleft G$ are <u>close</u> if $\forall$ finite $S \subset X$, and finite $A \subset G$, $\exists S' \subset X'$ and $S \leftrightarrow S'$ st

$$d(s_1, g s_2) \sim d(s_1', g s_2')$$
for $g \in A$.

Def 1: Θ_n is the space of equivalence classes of free actions of F_n on metric (simplicial trees).

Action of $Out(F_n)$ on $\Theta_n \Rightarrow \rho: F_n \to Isom(T)$

$\varphi \in Out(F_n)$

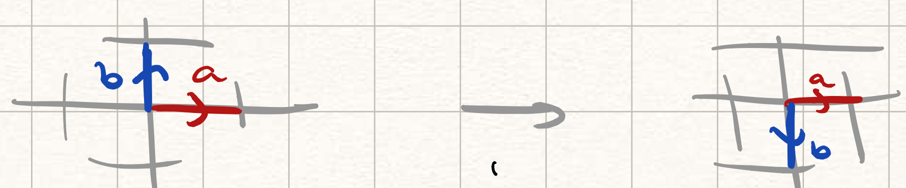
$\hat{\varphi} \in Aut(F_n)$

$$\begin{tikzcd}
F_n \arrow[r, "\rho"] & Isom(T) \\
F_n \arrow[u, "\hat{\varphi}"] \arrow[ur, "\rho \circ \hat{\varphi}"'] &
\end{tikzcd}$$

Two actions are equivalent if:

$F_n \curvearrowright T$, $F_n \curvearrowright T'$

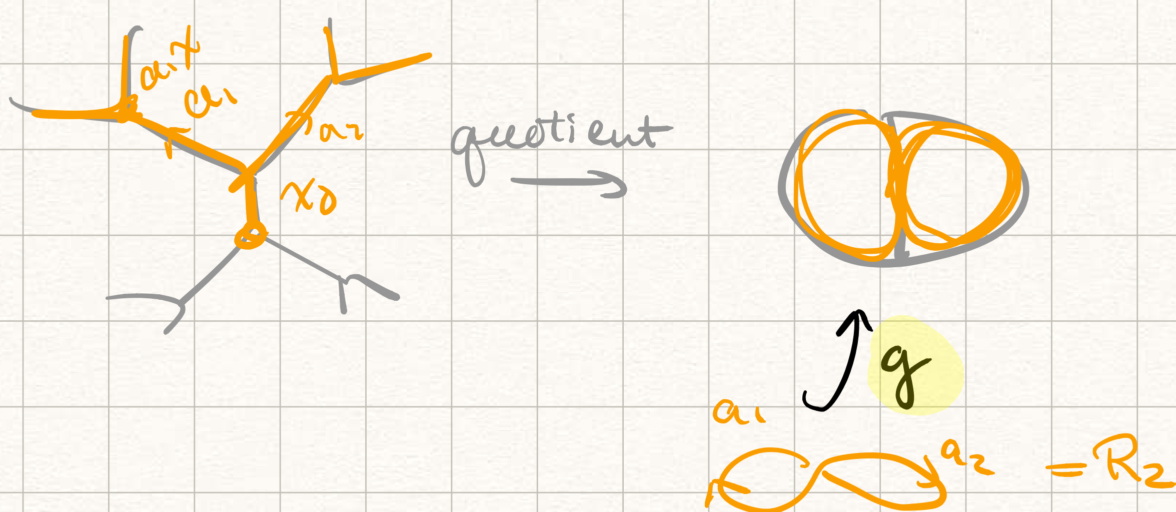
there's an isometry $T \to T'$ commuting w/ the action.



Def 2 (historically first def)

$F_n \curvearrowright T$. Quotient T/F_n is a graph G w/ $\pi_1(G) \cong F_n$.

Action is minimal $\Rightarrow$ G finite, has no univalent orbit vertices.



exe: g is a homotopy equiv.

$$g_*: \underset{\overset{\shortparallel}{F_n}}{\pi_1 R_n} \xrightarrow{\cong} \pi_1 G$$

g_* is a <u>marking</u>.

equivariant actions $\longrightarrow$ isometric graphs

$$\begin{tikzcd} G \arrow[rr, "\text{isometry}"] & & G' \\ & R_n \arrow[ul, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$$

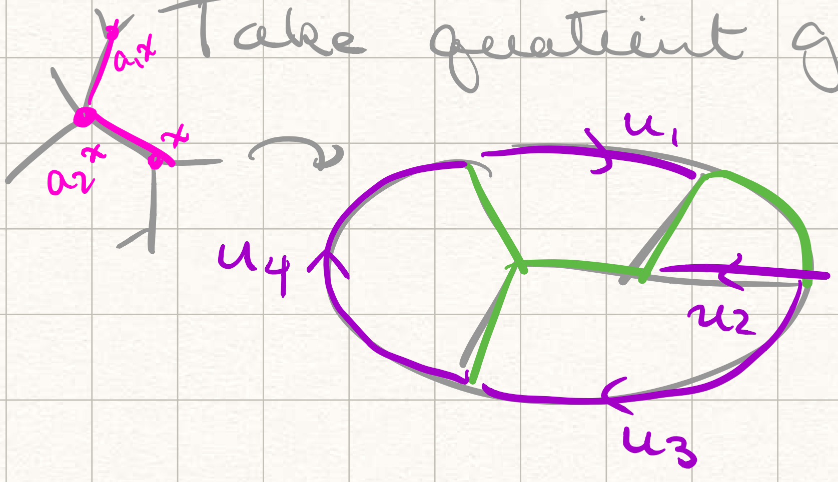
up to homotopy.

O_n is the space of equiv classes of metric marked graphs (G, g) st
- G is finite,
- has no 1- or 2-valent vertices,

$$g: R_n \xrightarrow{\simeq} G$$
$$(g_*: F_n \xrightarrow{\cong} \pi_1 G)$$

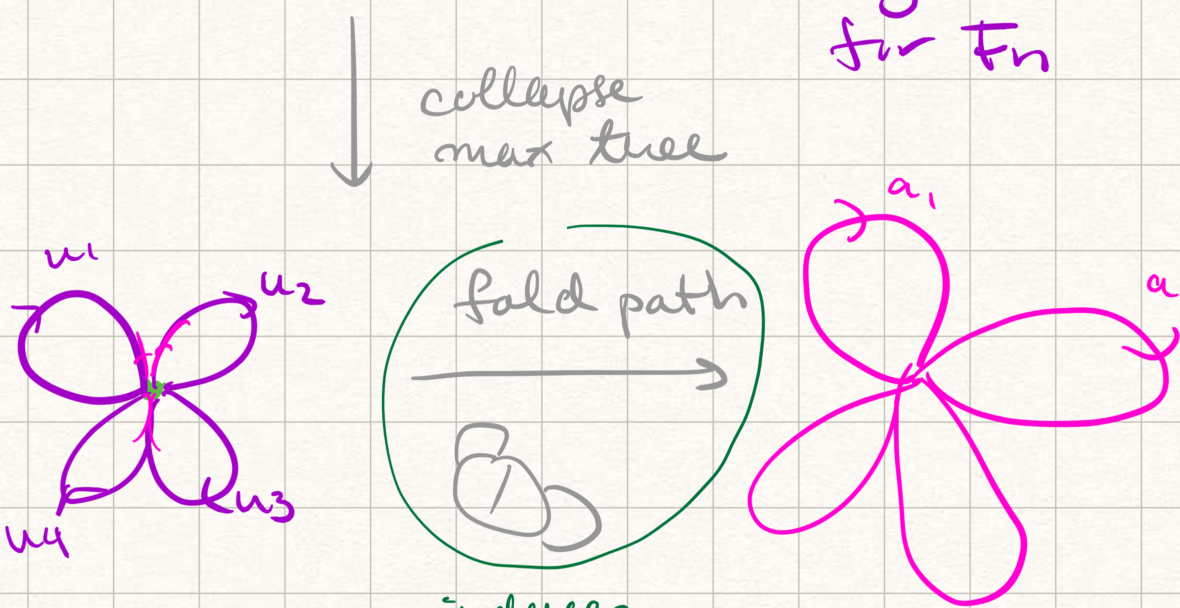
pict of marking:

Take quotient graph:



max tree

mark the rest of edges by a basis for F_n



induces homotopies

$\Rightarrow$ $\mathcal{O}_n$ is connected.

N.B. Only problem is that we have metric tree, so need rectify edge-len a bit.

ie: Normalize graphs st Σ edge-len $= 1$.

Thm: Θ_n (graph descript) decomposes into a disj. union of open simplices.

$\sigma(G,g)$ = simplex containing (G,g)
(G',g) obtained from (G,g) by varying edge lengths

ex:

describes varying the edge lengths

corresponds to collapsing n completely, (vertex NOT in simplex).

ex: