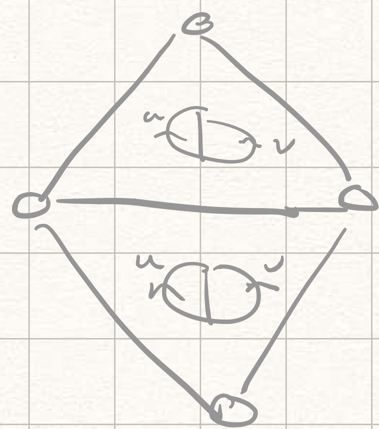
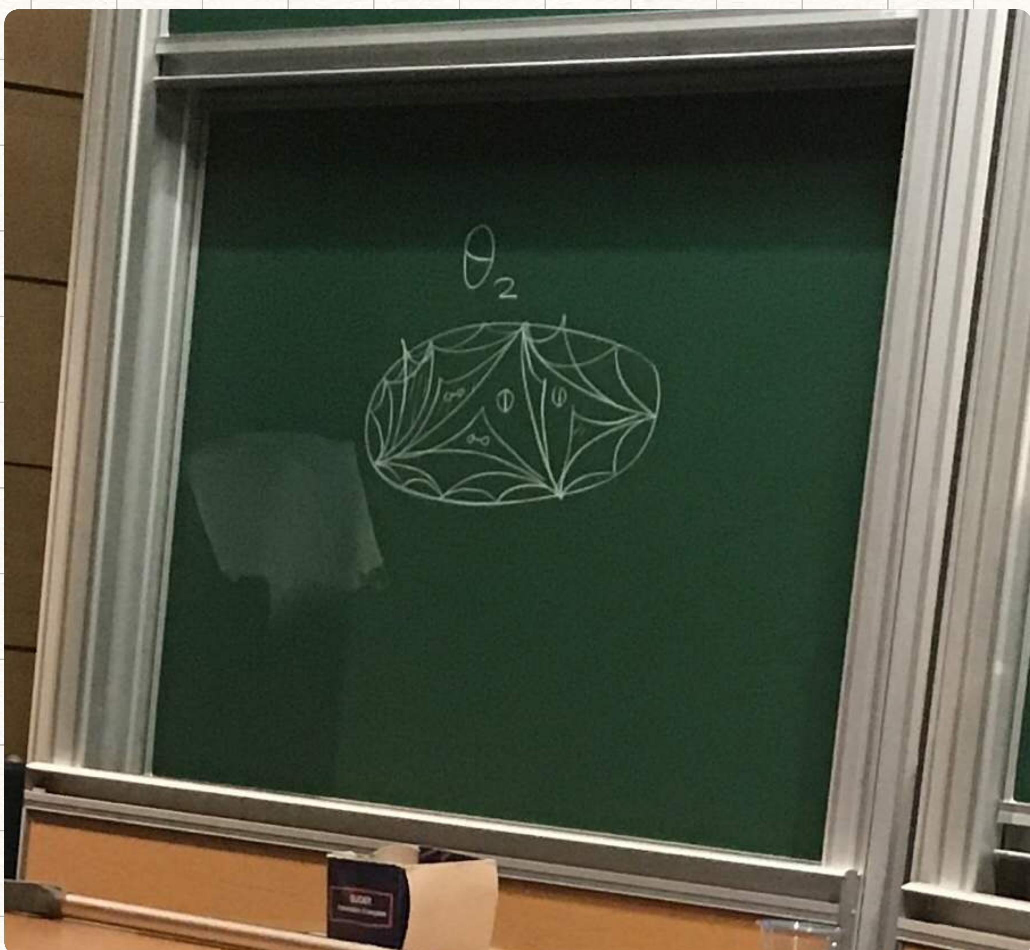
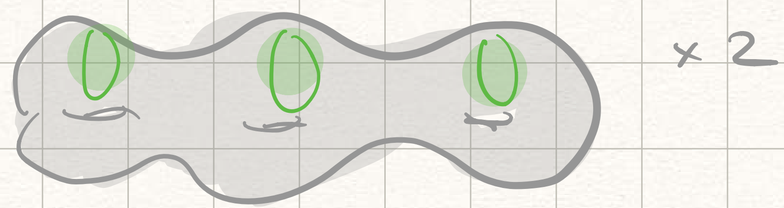
$$\Theta_n = \coprod o(G,g) \,/\, \text{face relations}$$





Using $M_n = \#_n S^1 \times S^2$



OR :

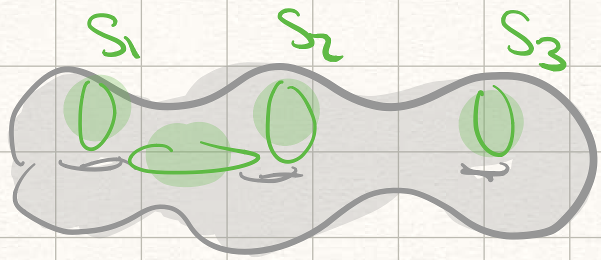
A sphere system in M_n is a set of embedded 2-spheres which can be isotoped to be disjoint.

- S_i doesn't bound a ball
- S_i not isotopic to S_j.

analogy: curve complex.

$S(M_n)$ = sphere complex :

- vertex for each sphere
- k-simplex for each system of $(k+1)$ spheres.



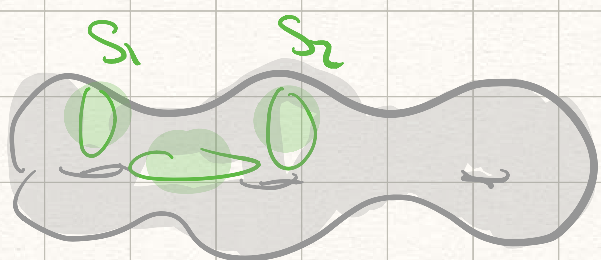
ex of sphere system.

M_n is a set $S = \{s_1, \ldots, s_k\}$

Def: A sphere sys. is complete if it cuts M_n into simply connected pieces. (each is $S^3 - \coprod_k B^3$)

exc: Takes at least n-spheres to cut, and can't fit more than $3n-3$ spheres.

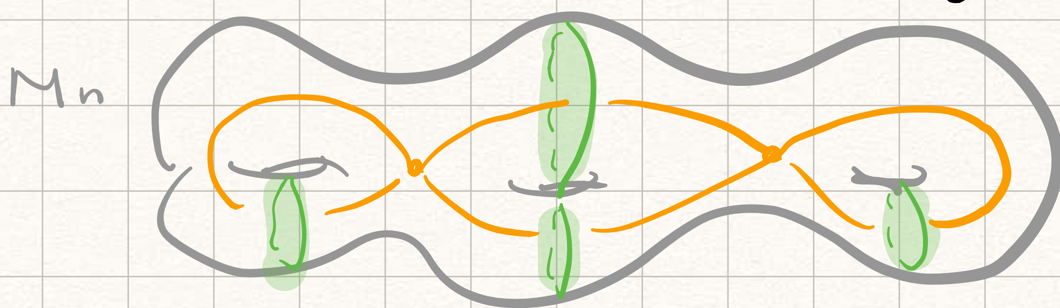
Let $S_\infty(M_n)$ be the subcomplex formed by an incomplete system.



Def 3: $\mathcal{O}_n = S(M_n) - S_\infty(M_n)$
= union of open simplices corresponding to complete sys.

Showing: $D_2 \Leftrightarrow D_3$

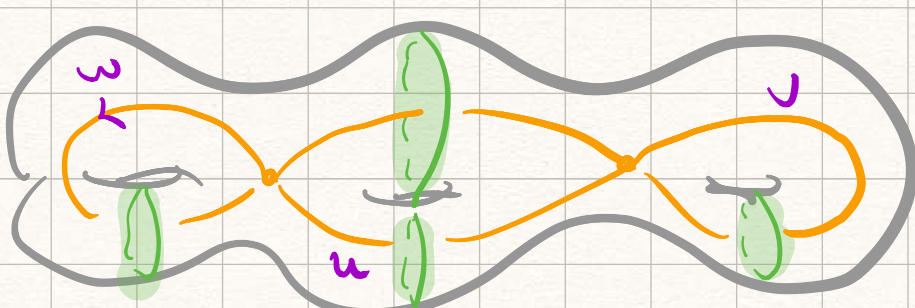
Let $S = \{S_1, \dots, S_k\}$ complete sphere sys $\subseteq M_n$.



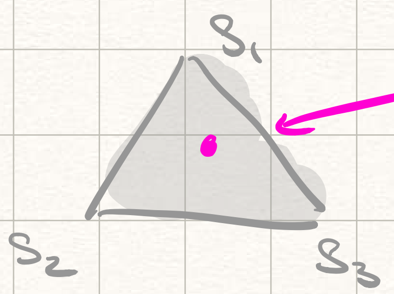
Identify $F_n \equiv \pi_1 M_n$

$G(S)$ = dual graph $\subseteq M_n$ marks $G(S)$

$F_n \xrightarrow{\cong} G(S)$

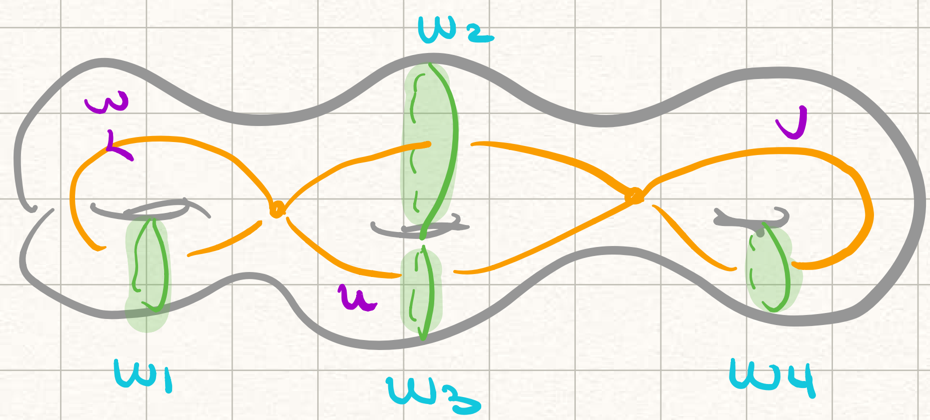


Simplex : $S = \{s_1, \dots, s_k\}$

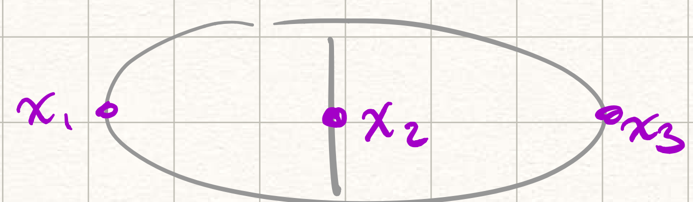


pt in simplex
= weight sphere system.

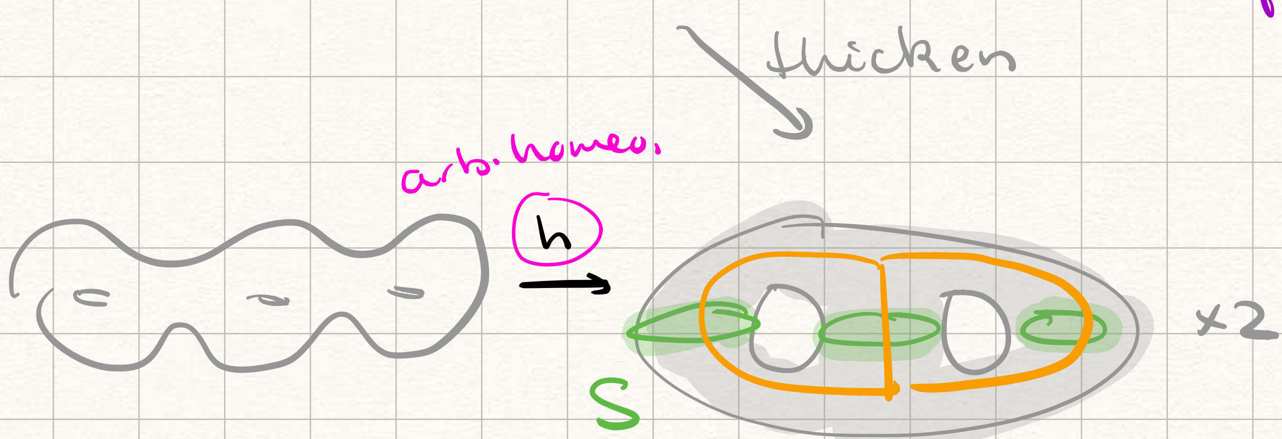
w_i = weights on sphere



Given (G, g) a marked graph



put a pt x_e in the middle of e



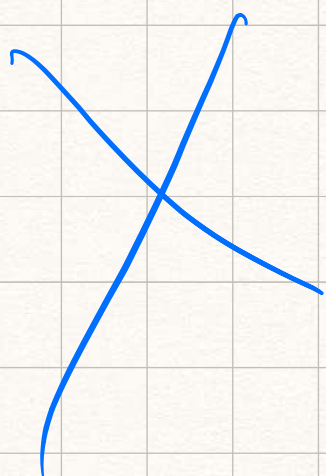
$h^{-1}(S)$ is a sphere system on M_n w/ dual graph G.

NB: May not get right marking

$h^{-1}(S) \longleftrightarrow (G, f)$,

f is some F_n,

$$\begin{tikzcd} F_n \arrow[r, "f"] \arrow[d, "fg^{-1}"'] & \pi_1 G \\ F_n \arrow[ur, "g"'] & \end{tikzcd}$$



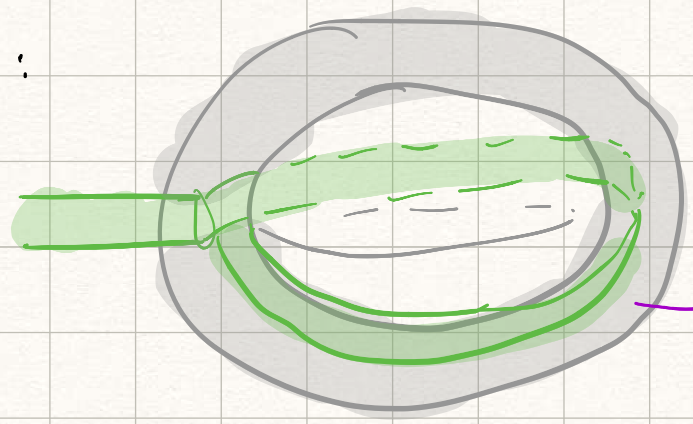
Recall:

Dehn twist

$1 \longrightarrow DT \longrightarrow \pi_0 \,\mathrm{Homeo}(M_n) \longrightarrow \mathrm{Out}(F_n) \longrightarrow 1$

Claim: A Dehn twist acts trivially on $S(M_n)$

Idea:



$\Rightarrow$ action was trivial.

$S = \{s_1, \dots, s_k\}$

make the S_i transverse to twisting sphere.

$\Rightarrow$ DT acts trivially. $\square$

Thm: Θ_n is contractible & $Out(F_n)$ acts properly.

Pf: (1) (Culler + Speaker) Using graphs & combinatorial Morse theory.

(2) (Skora) Trees, and holding pads.

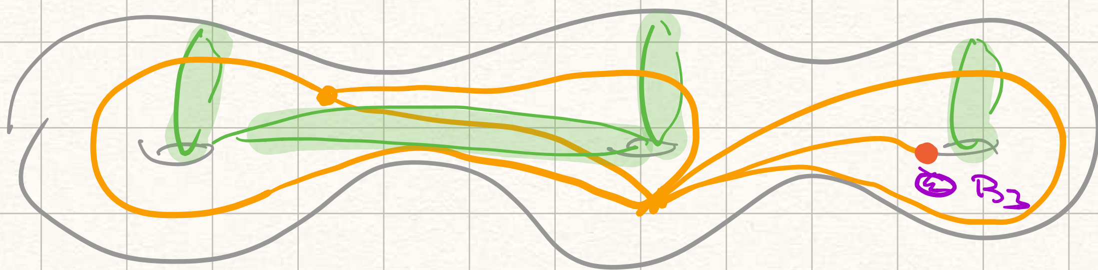
(3) (Hatcher) Sphere sys + surgery paths.

Also shows sphere sys contractible.

NB: There are notes for this course on Open Math Notes.

Q: Talked about $\mathrm{Out}(F_n)$, but what about $\mathrm{Aut}(F_n)$?

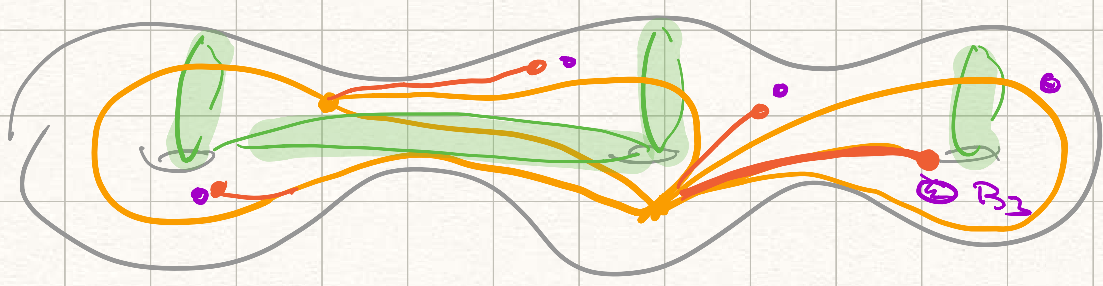
$M_n - B_3$



Look at sphere systems in M_n, get $S(M_{n,1})$ sphere complex, $\mathrm{Aut}(F_n)$ acts on $S(M_{n,1})$.

Same pf shows $S(M_{n,\cancel{1}^{s}})$ contract.
So is $S(M_{n,\cancel{1}^{s}}) - S_\infty(M_{n,\cancel{1}^{s}}) = \mathcal{O}_{n,\cancel{1}^{s}}$.

in general: $M_{n,\underline{s}} = (M_n - \coprod_s B^3)$



& get:

$$1 \longrightarrow DT \longrightarrow \pi_0 \mathrm{Homeo}(M_{n,s}, \partial M_{n,s}) \to A_{n,s} \to 1 .$$

In terms of graphs:

dual to sphere sys is a graph w/ leaves.

$$A_{n,s} = \pi_0(HE(R_{n,s}, \partial R_{n,s}))$$