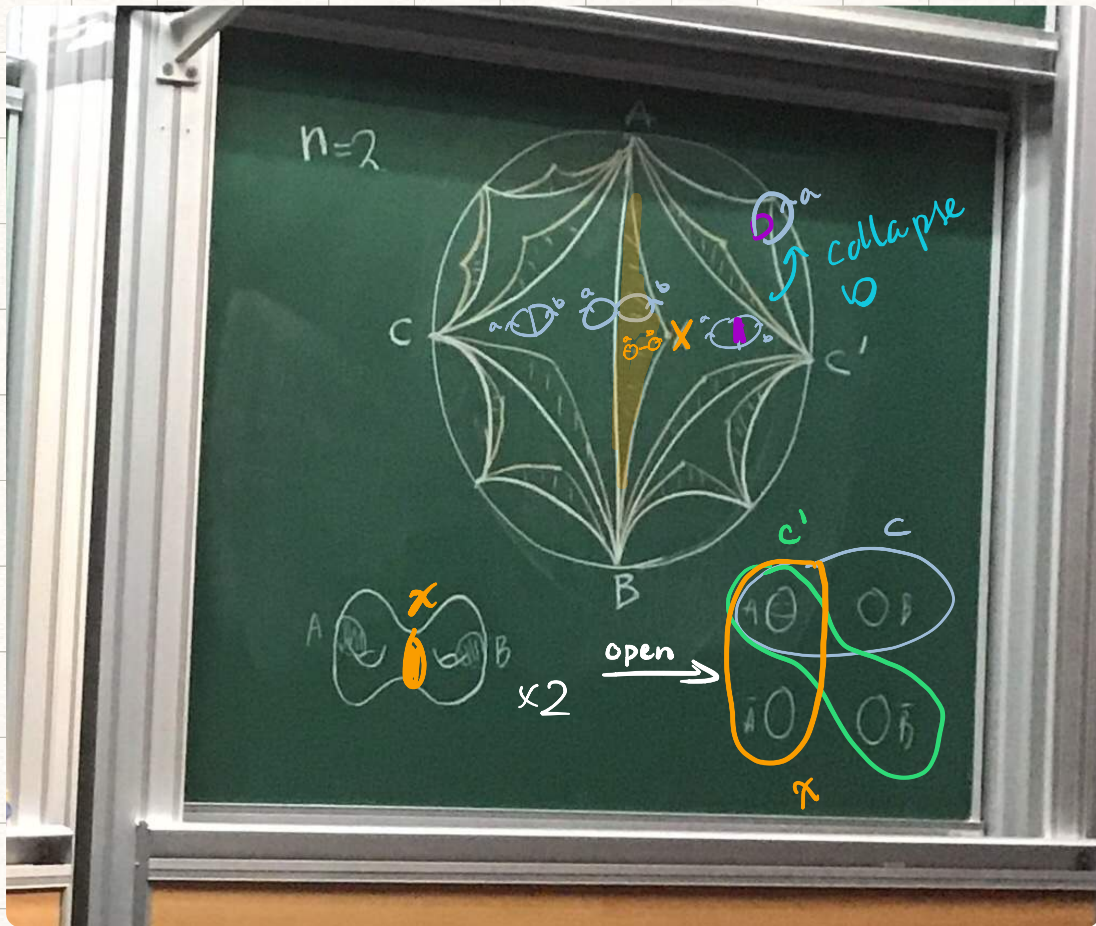


We define $\mathcal{O}_n$, $n \geq 2$ (3 ways)

And variants $\mathcal{O}_{n,s}$ (2 ways)

Sphere complexes:

$$M_{n,s} = \# S^1 \times S^2 - \bigsqcup_s B^3$$

$S(M_{n,s})$ = complex of isotopy classes of spheres

$\cup$

$\mathcal{O}_{n,s}$ = subspace of complete sphere systems.

$\parallel$

$S(M_{n,s}) - S_\infty(M_{n,s})$

$F_n = \pi_1(M_1) * t$

$F_n = \pi_1 L \underset{1}{*} \pi_1 R$

Rmk on $S(M_n, 0)$

Also called the "free splitting complex of F_n".

2012: Handel-Mosher: $S(M_n)$ is Gromov hyperbolic.

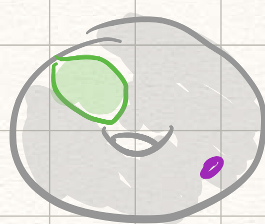
2013: Hilion-Horbez gave a pf using spheres.

cool. Analogue curve complex.

Rmk on $O_{n,s}$ for $s>0$.

① Makes sense for $n=1$.

they're just statements



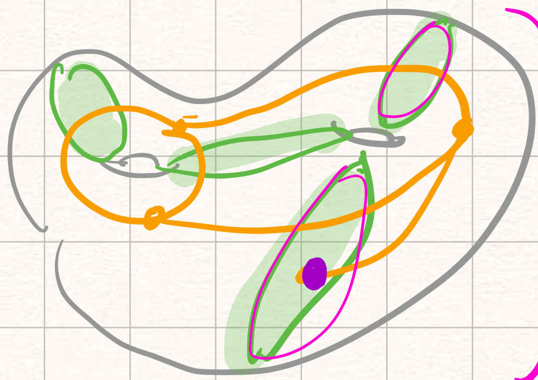
$S^1 \times S^2$

double solid torus w/ punct.

② Makes sense for $n=0$, if $s \geq 4$, $M_0 = S^2$

3-sphere w/ zero handles attached, 4 punct.

$S(M_{n,s})$ for $s>0$: don't use spheres parallel to 2 components.



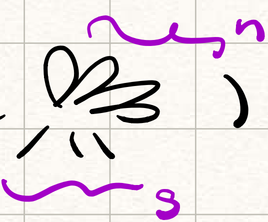
trivial sphere = spheres that don't bound balls,

if didn't have holes, these two spheres are equiv.

still sphere sys w/ weights, but punctures (leaves •) don't have lengths/weights.

$$A_{n,s} = \pi_0(\text{Homeo}\, M_{n,s})/_{DT}$$

$$= \pi_0\, \text{CHE}\,(\ \ )$$



n

s

generalization of Out Fn

$s=0$

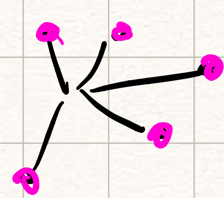
$\Rightarrow$ Out Fn.

$S(M_{n,s})/A_{n,s}$ $(n \geq 1)$ is "Moduli Space of tropical curves".

$$(\cong \mathcal{C}(S_{g,s})/A_{2g,s})$$

curve complexe

If $n=0$,

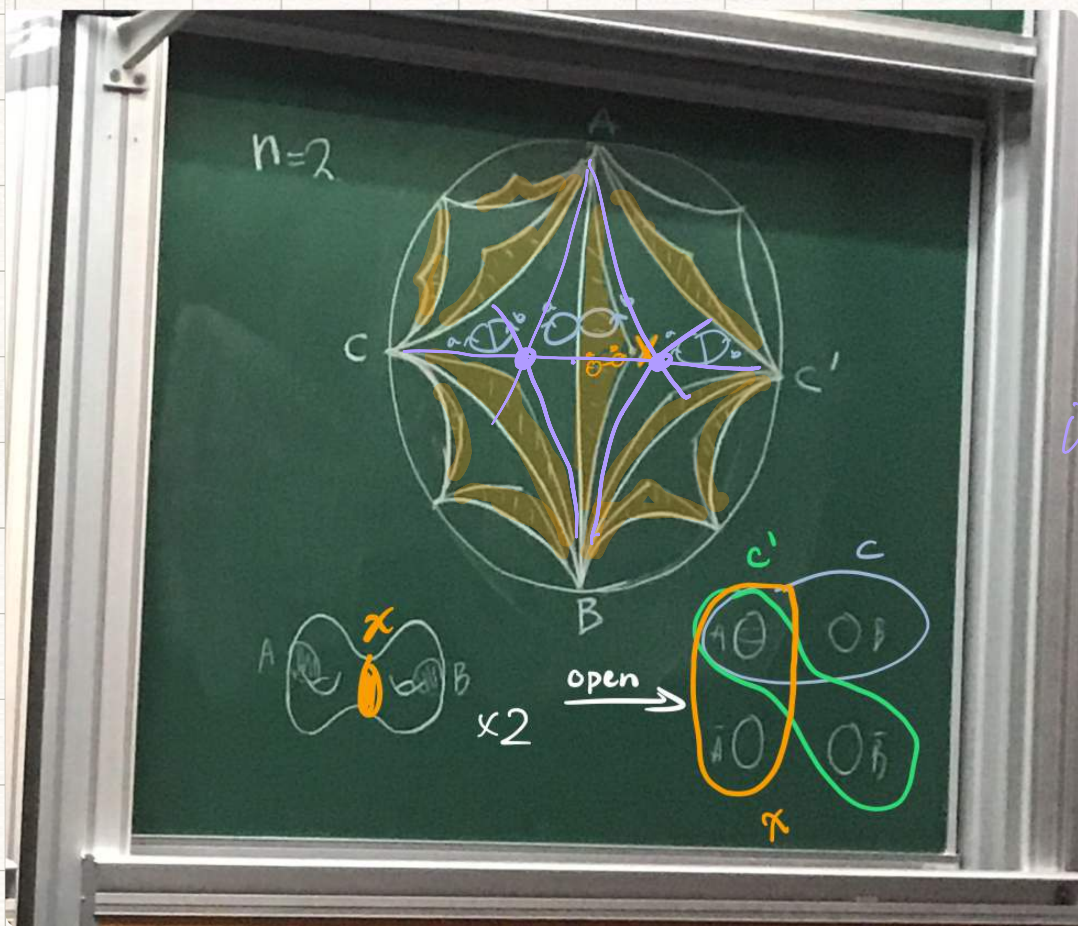


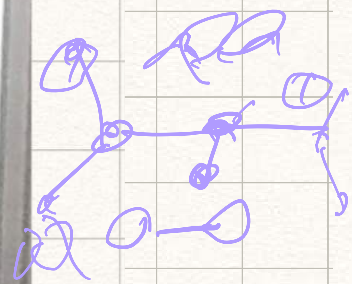
s

$A_{0,s}=1$

have to fix leaves.

$$S(M_{0,s}) = SC(M_{0,s})/A_{0,s}$$
$$= \text{"tropical Grassmaniann"}$$
$$\simeq \vee S^{s-4} \text{ (spheres)}$$





orange = barbell

To fix the fact that $\Theta_{n,s}/A_{n,s}$ is not compact $(n\geq 2)$, + not qi to grp, not sure if this goes thm

↳ next: will want nice space to study.

For $n\geq 2$,

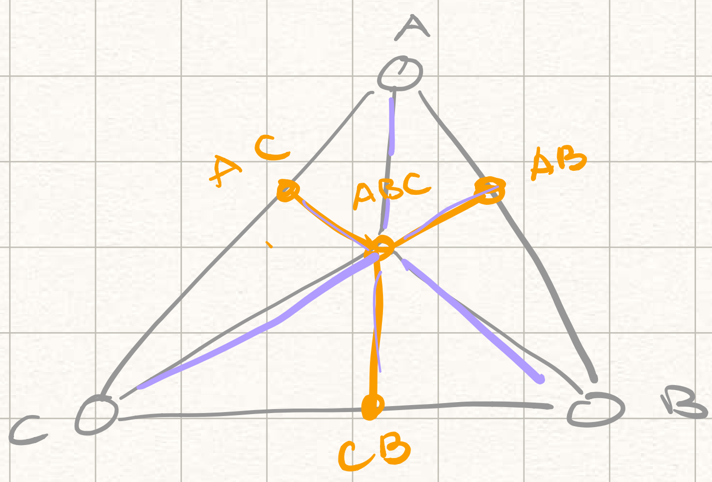
take $S'(M_{n,s})$ = barycentric subdivision of $SC(M_{n,s})$

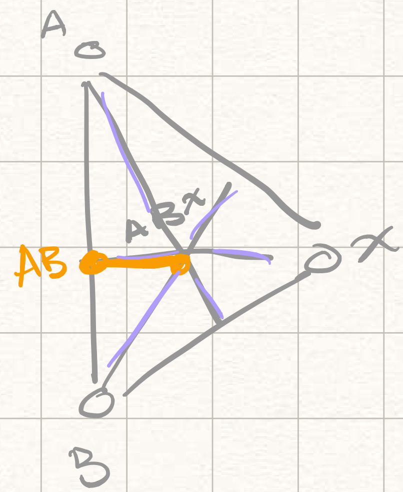
$\cup$

$K_{n,s}$ = subcomplex spanned by simplices

but throw out pts not in ont = empty verts + edges that collapse loop.

vertex = chain $\sigma_0 \subset \cdots \subset \sigma_k$.





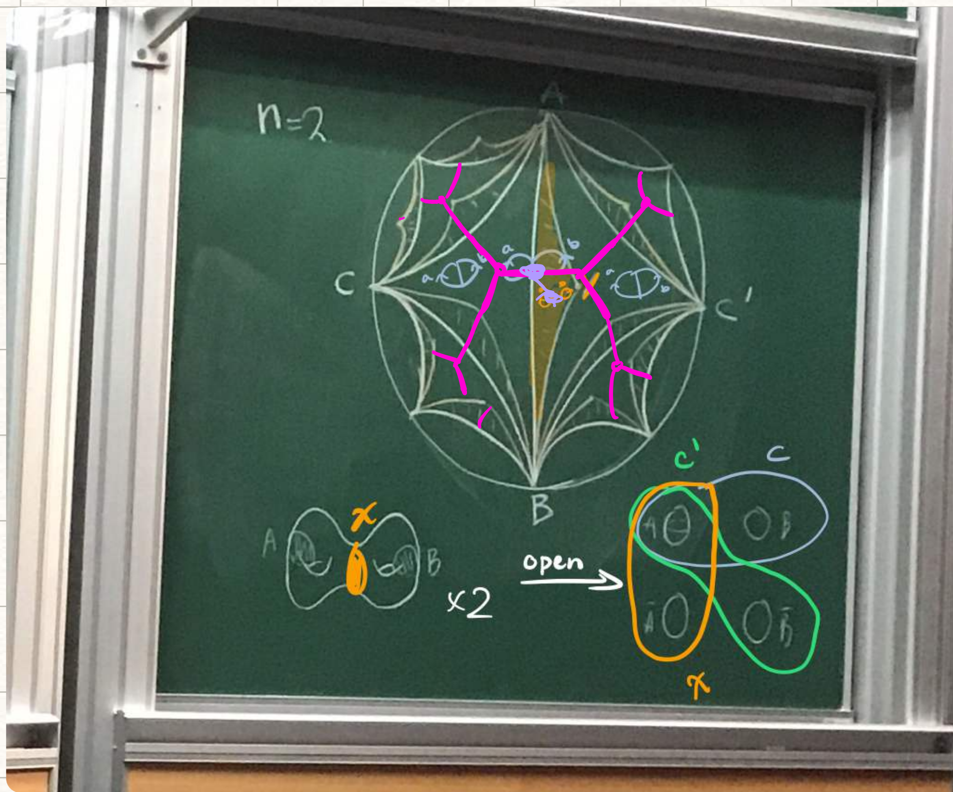
$K_{n,s}$ = spine of $O_{n,s} \subset O_{n,s}$
but misses "bad systems".
(vertices not in Outer space)

Action of $A_{n,s}$ is proper on $K_{n,s}$
and $K_{n,s}/A_{n,s}$ is compact.
and $O_{n,s} \searrow K_{n,s}$ as deform. retract.

exc: $\dim O_{n,s} = 3n-4+s \quad (n \geq 2)$

$\dim K_{n,s} = 2n-3+s \quad (n \geq 2)$

Local structure of $K_{n,s}$



K = simplex in $K_{n,s}$ in barycentric subdivision

= chain of systems (of simplices)

$S_0 \subset S_1 \subset \cdots \subset S_k$ all complete

= chain of collapses

$$G_0 \leftarrow \cdots \leftarrow G_{k-1} \xleftarrow{c} G_k$$

+ markings which commute

R_n

(c collapses forests)

i.e collapsing edge in 2D.

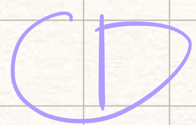
max degree
can't expand
↑ only
collapse

link of a vertex: G trivalent,

link of G = graphs obtained by collapsing forests

= "geometric realization of the partially ordered set of forests in G." (just called this).

marking graph



ex: "link of rose" = minimal complete sphere sys S_0

→ can only expand.

$M - S_0$

$\wr\wr$

$M_{0,2n}$: link$(S_0) \cong S(M_{0,2n})$

ball w/ 2n holes.

$\simeq \bigvee S^{2n-4}$

homotopy eq.

cut this sphere sys open.

since it's min, look at spheres can add to it

$$\text{link}(S_0) \cong S(M_{0,2n}) \simeq \bigvee S^{2n-4}$$

if G not trivalent, not a rose
then $\underset{\text{sphere sys.}}{S}$ not minimal, not maximal

$$SC(M_{n,s}-S) \underbrace{*}_{\text{simplicial join}} F(G) = \text{Forest in } G$$

wedge of spheres

Thm: $F(G) \simeq \vee S^d$
$\Rightarrow \text{link}(\Delta) \simeq \vee S^D$
where $D = \dim K_{n,s} - 1$.

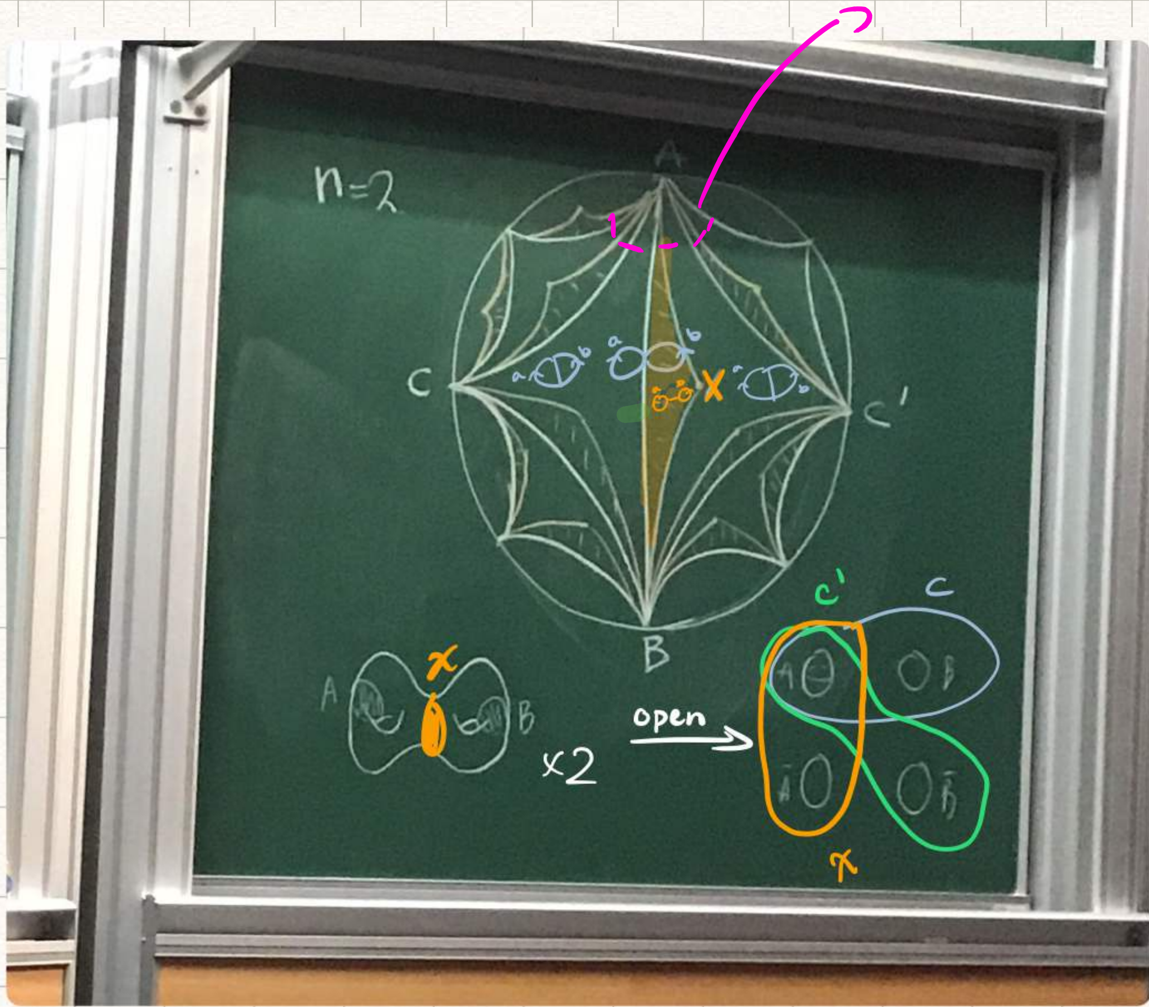
One more space:

$bO_{n,s}$ "bordification"
another way to get proper

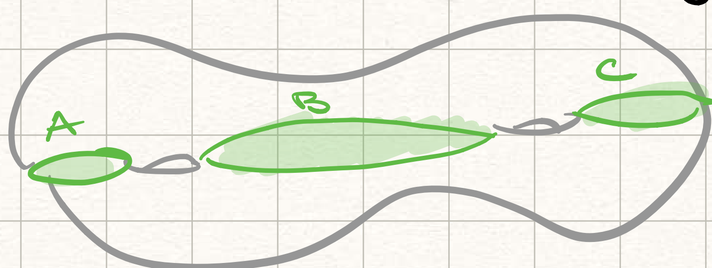
① $SC(M_{n,s})$ = subcomplex spanned by non-sep spheres.

Given $\overset{\substack{\text{simplex corresponds to } S \\ \shortparallel}}{\sigma(S)} \subset S^{red}(M_{n,s})$, define $J(S) \subset \sigma(S)$ by sliding off faces of $\sigma(S)$ that are opposite to **core** subgraphs.

cut off bad vertices.



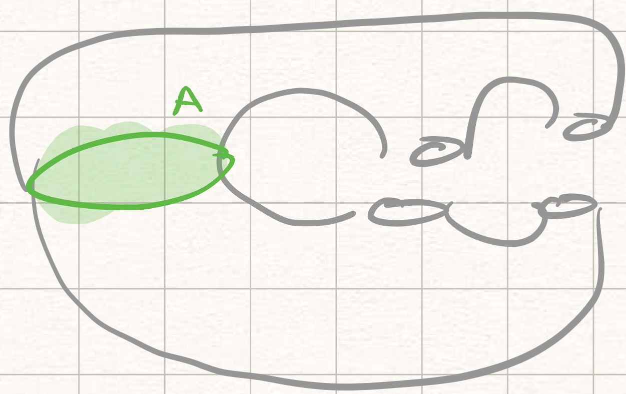
Def: $S_0 \subset S$ complete sys.



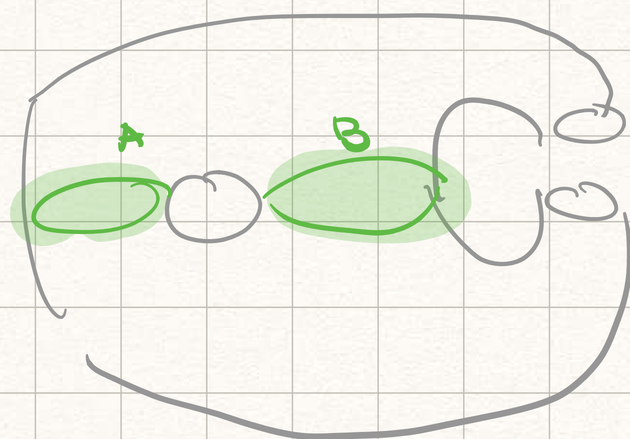
Let $S_1 = S - S_0$, S_0 is core if spheres of S_0 are <u>non-sep.</u> in $M - S_1$

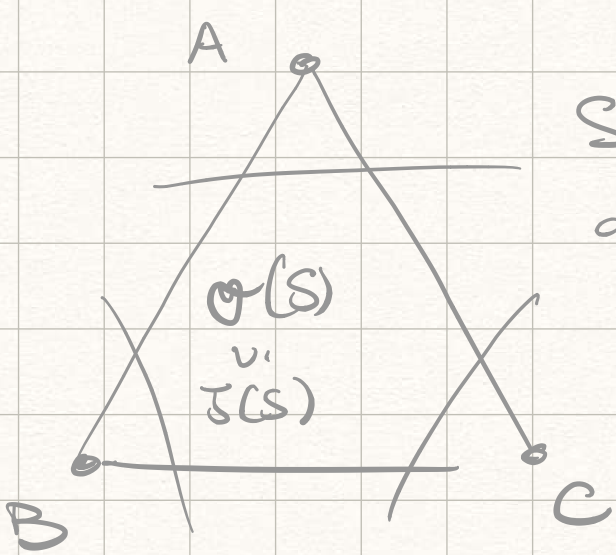
ex: $S_0 = \{a\}$, $S_1 = \{b, c\}$,
NOT core.

$M - S_1$

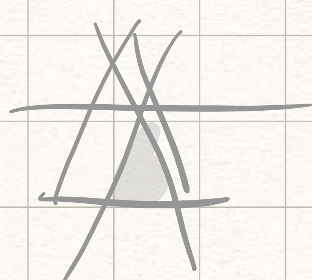


$S_0 = \{a, b\}$, $S_1 = \{c\}$, $M - S_1$ is core.





Slice off faces opposite core subgraphs, slice deeper if S_0 smaller.



$T(S)$ homeo to bordification.