Boundaries of Hyp. Spaces

Boundary of a hyp. grp

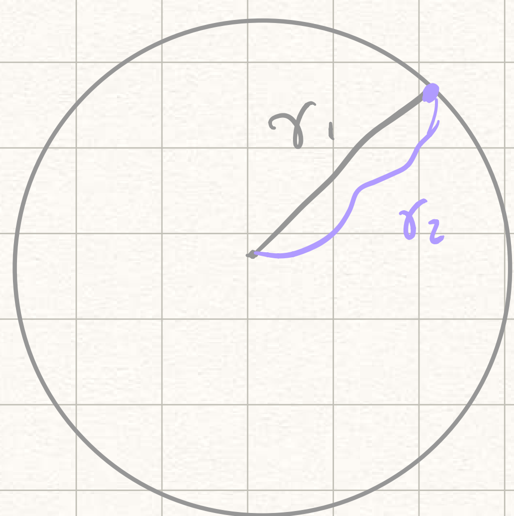
- canonical top. object

Q: what does it tell you?

- Rel. hyp, $\partial(G,P)$.

X proper geod. metric space:

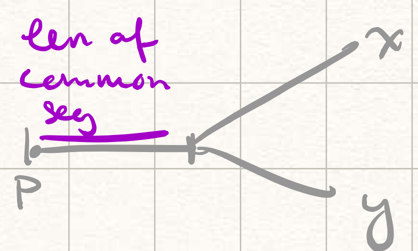
Def 1: ∂X is the set of geod ray $x_0/\sim$ where $\sim$ is bounded Haus. dist, ie, $\gamma_1 \sim \gamma_2$ if $\operatorname{im}\gamma_1 \subset N_c(\operatorname{im}\gamma_2)$
$\operatorname{im}\gamma_2 \subset N_c(\operatorname{im}\gamma_1)$



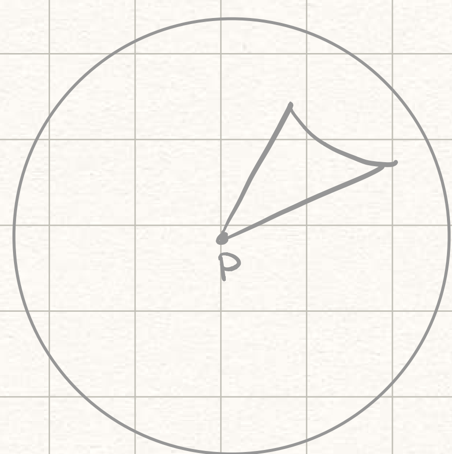
Def 2: quasi-geod rays at some $x_0 \in X$,
mod $\sim$ bounded Hausdorff dist.

Def 3: "Sequences tending to ∞"
Gromov prod:
$$(x,y)_p := \frac{1}{2}\left(d(x,p)+d(y,p)-d(x,y)\right)$$

in tree:



in hyp:



? relisten.

Say (a_i) "tends to ∞" if
$$\lim_{\substack{i\to\infty \\ j\to\infty}} \underbrace{(a_i, a_j)}_{\text{Gromov prod}} = \infty$$

∂X = set of seq tending to ∞, where $(a_i) \sim (b_i)$ if $\lim_{\substack{i\to\infty \\ j\to\infty}} (a_i, b_j) = \infty$.

Now put top: extend Gromov prod to ∂X: $m, n \in \partial X$,

$$(m,n)_w = \sup_{\substack{x_i \to m \\ y_i \to n}} \{ \liminf (x_i, y_j)_w \}$$

eq classes

Put top on $\hat{X} = X \sqcup \partial X$,

$x \in X$, $B_r(x)$ metric ball

$x \in \partial X$, $N_r(x) = \{ y : (x,y)_w > r \}$

if $y \in \partial X$ above

if $y \in X$ then use

$$(x,y)_w = \sup_{x_i \to x} (\liminf (x_i, y)_w)$$

Def: X is δ-hyp. $\iff$

$\forall x, y, z, w$,

$$(x, y)_w \geq \min\{(x,z)_w, (y,z)_w\} - \delta$$

ie "thin rectangle" (see exc).

In δ-hyp space (w/ thin Δs),

$$(x,y)_p \leq d(p, [x,y]) \leq (x,y)_p + \delta.$$

X proper $\Rightarrow$ $\hat{X} + \partial X$ are compact.
∂X Hausdorff.

Thm: X is q-isom to X', then
$\partial X \to \partial X'$ is a homeo.

Let $f: X \to X'$ be q-isom w/ q-inverse g
$m \in \partial X$ rep'd by q. geod ray γ,
$f(\gamma)$ is also q-geod ray,

$$\partial f([\gamma]) = [f(\gamma)] \in \partial X$$

$d_H(\gamma_1, \gamma_2) < K$, then
$d_H(f(\gamma_1), f(\gamma_2)) \leq \lambda K + C$
$d_H(g \circ f(\gamma), \gamma)$ is bounded.

pre-im of open is open:
$\bar{y} \in N_R(\bar{x}) \Rightarrow \partial f(\bar{y}) \in N_n(\partial f \bar{x})$
where $x_i \to \bar{x}$, $y_i \to \bar{y}$

$(f(x_i), f(y_i))_{f(x_0)}$
$\geq d(f(x_0), [f(x_i), f(y_i)]) - \delta$
$= d(f(x_0), t') - \delta$ for some t' on
$[f(x_i), f(y_i)]$

$\geq d(f(x_0), f(t)) - K - \delta$ for some $t \in [x_i, y_i]$

$\Rightarrow$ this map is q-geod.

$\geq \frac{1}{\lambda} d(x_0, t) - C - K - \delta$

$\geq \frac{1}{\lambda} d(x_0, [x_i, y_i]) - C - K - \delta$

$\geq \frac{1}{\lambda} (x_i, y_i)_{x_0} - (C + K + \delta)$

if $(x_i, y_i)_{x_0} \geq \lambda(n + C + K + \delta)$ then

$$\frac{1}{\lambda}(x_i, y_i)_{x_0} > n + C + K + \delta$$

$\Rightarrow \frac{1}{\lambda}(x_i, y_i)_{x_0} - (C + K + \delta) > n$

$\Rightarrow$ the map is cont $\square$

Now ∂G, $G \curvearrowright_{\text{geom}} X$ prop. hyp. space.

$\partial G = \partial X$.

Pt: this can tell you a lot!

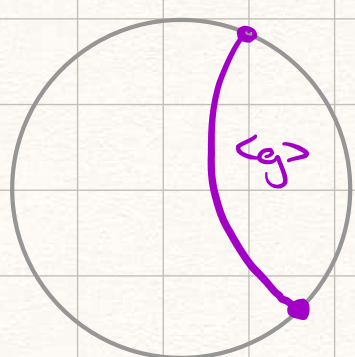
$\partial \mathbb{F}_2 = \mathcal{C}$ a Cantor set.

Thm: if G is hyp & $\partial G = \mathcal{C}$, then G is virt. free.

Thm: $\partial(\Sigma_g) = S^1 = \partial \mathbb{H}^2$
if $\partial G = S^1$, then G is virtually a surface grp.

Problem: Take your fav. ∂M.
- which G have $\partial G \cong M$?
- Which M is the ∂G of G hyp?

Geometry of subgrps



$\partial \langle g \rangle = 2$ pts

Def: A subset A of a geod metric space (X, δ) is ε-quasi-convex if any geodesic $[a_1, a_2]$ in X b/w pts of A is in an ε-nbhd of A.

A an infinite subgrp is q-convex if it is q-convex for $A < G$ fg in $\mathrm{Cay}(G, S)$, S some finite gen. set.

G hyp. doesn't depend on S.

A q-convex $\Leftrightarrow$ orbit of A, Ax_0, is q-convex in X, $G \curvearrowright_{\text{geom}} X$

$A < G$ hyp,

$Ax_0 \subset X$, $x_i \to \infty$

$\Lambda A = \{ [(x_i)], x_i \in Ax_0 \}$

Def: X proper hyp $N \subset \partial X$,

$C(N) := \bigcup_{a,b \in N} (a,b)$ is

the convex hull of N in X.

$\left(\begin{array}{l} \text{Let } a, b \in \partial X, \exists \text{ a bi-infinite geod} \\ \quad \gamma : (-\infty, \infty) \to X \\ \quad \lim_{i \to \infty} \gamma(-i) = a, \quad \lim_{i \to \infty} \gamma(i) = b \\ \\ \quad \text{call this } (a, b) \end{array} \right)$

Prop: $G \overset{\text{geom}}{\curvearrowright} X$

- quasi-convex subgrps of G are hyperbolic.

- $A < G$ is q-convex
  $\iff C(\Lambda A)/A$ has finite diam

- $A \hookrightarrow G$ is q-convex,
  $\Lambda A = \partial A \hookrightarrow \partial G$.