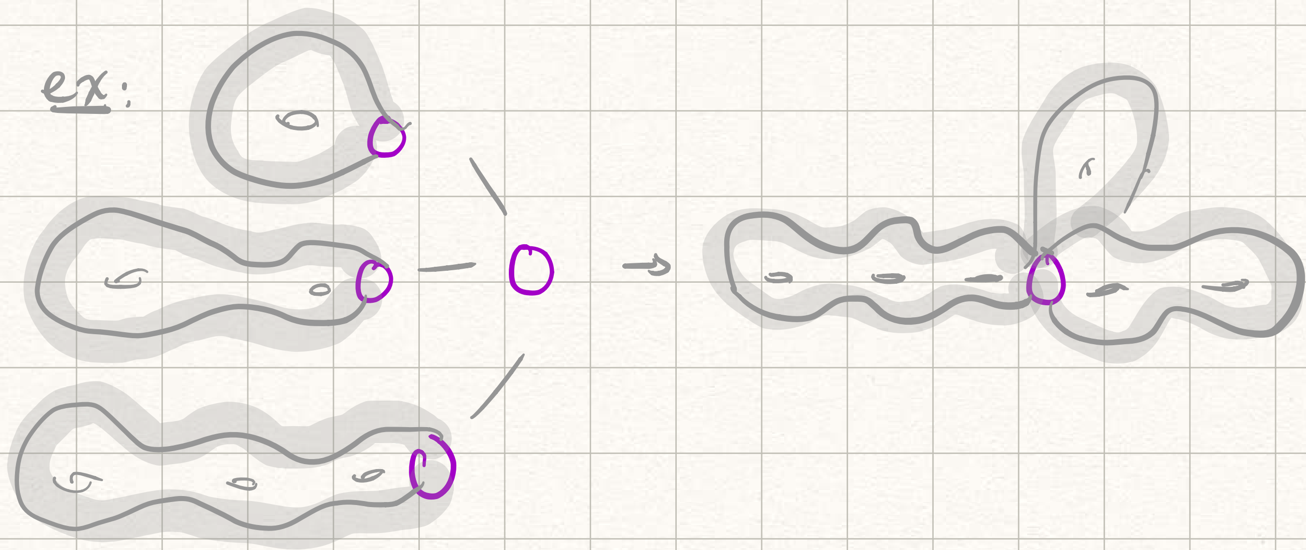
Note: The convex hull is q-convex.

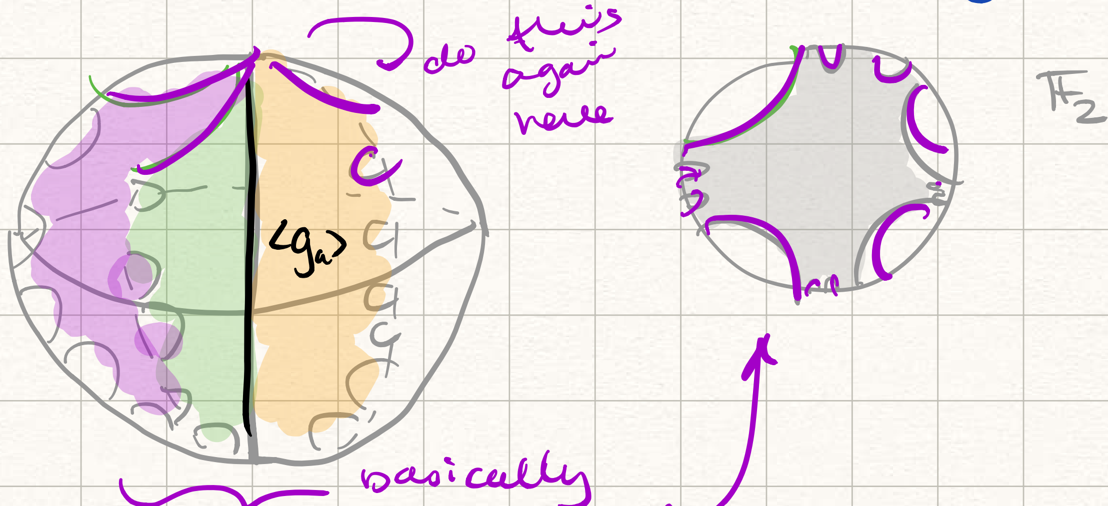
Bowditch: See a lot of q-convex & how they are put together.

ex:



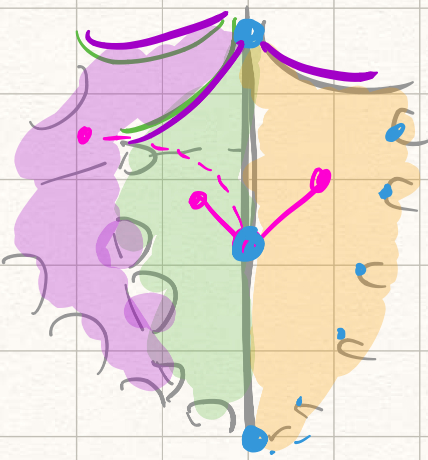
Thicken up in $\mathbb{R}^3$
$\Rightarrow$ hyp. 3-manifold

Q: What does the boundary look like?



do this again here

basically 3 sheets of (like a book)



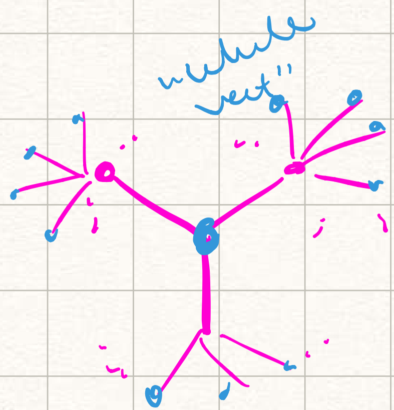
- Stab by two-ended grp

val 3

- Stab by free grp

∞-val

Bass-Serre tree of splitting.



$$\begin{array}{ccc} \mathbb{F}_4 & — \; \mathbb{Z} \; — & \mathbb{F}_2 \\ & | & \\ & \mathbb{F}_6 & \end{array} \Bigg\} \text{ graph of grps.}$$

Def (Elementary splitting)

Minimal co-finite action on (no proper inv sub-tree), a simplicial bipartite tree

$$(V_e \sqcup V_{ne}, \mathcal{E})$$

- V_e vertices - stabilized by 2-ended grps.
- V_{ne} vertices - some other stab, not two ended.

Bowditch: Γ, 1-ended hyp.

$\partial\Gamma \neq S^1$,

look at $\partial\Gamma$ to get elmentary splitting; $\partial\Gamma \setminus \{x\}$ has more than 1 end.

local cut pt of $\partial\Gamma$, $LC(\partial\Gamma)$

$x \in LC(\partial\Gamma)$,

val(x): # ends $\partial\Gamma \setminus \{x\}$

Bowditch $\sim$ on $LC(\partial\Gamma)$:

$x \sim y$ if val(x) = val(y) & $\{x,y\}$ is a cut pair.

Equiv. classes

- $|[x]| = 2$, stab is $\mathbb{Z}$, and these correspond to elmentary vertices.

- val(2) - ∞ many equiv. pts. $\overline{[x]}$ closure of Cantor set.

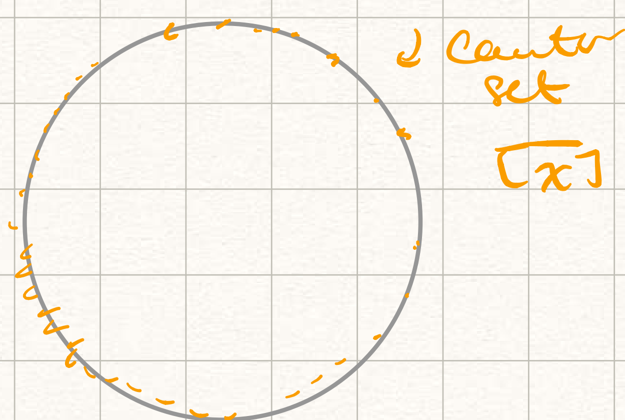
boundary of sheets
= Cantor set.

Type of equiv. classes:

- val$(x) \in [3, \infty)$ (vertices)
  $\#[x] = 2$
  (these are stabilized by 2-ended grps)

- val$(x) = 2$, $\#[x] = 2$ ( )

- val$(x) = 2$, $\#[x] = \infty$ ( )



Action preserves cyclic ord

Bowditch:

Γ 1-ended hyp $\partial\Gamma \neq S$,
$\exists$ canonical elmtary splitting,

$(V_\varepsilon \sqcup V_{N\varepsilon}, \varepsilon)$

- elementary (white) are two-ended equiv classes w/ $\#[w]=2$.
- stabilizers of vertices + edges are q-convex.
- non-elementary: either
  - cyclically hanging Fushian
  - rigid
  - don't split anymore.

Def: A rel. hyp. grp pair (G,P) is a grp pair which acts geometrically finitely on some proper hyp. met. space X.

Geometrically finite action on X:

$G \curvearrowright$ properly disc. & by isoms on X.

(i) Every elmt of ∂X is either:

- a canonical limit pt
- or bounded parab. pt

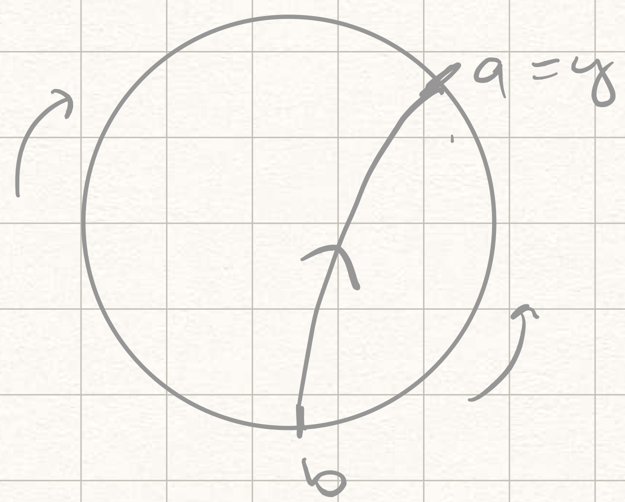
(ii) The elmts of $\mathcal{P}$ are exactly the max parab. subgrps.

P is parabolic: infinite, contains no loxo. & fixes $x_P \in \partial X$.

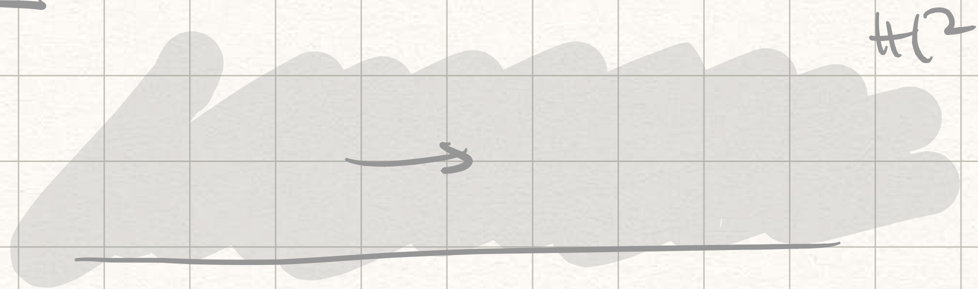
bounded if: $(\partial X \setminus \{x_P\}) / P$ is compact

$y \in \partial X$ is a canonical limit pt of G if $\exists$ seq $(g_i)_{i \in \mathbb{N}}$ & a, b st

$g_i(y) \to a$, $g_i(x) \to b$ if $x \neq y$, $x \in \partial X$.



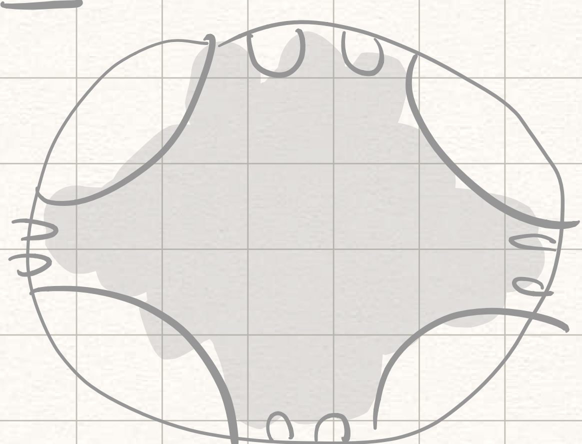
ex: $z \mapsto z+1$

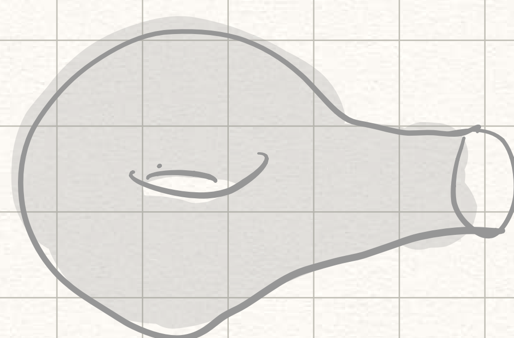


$\partial X - \{x_p\}$

$(\partial X - \{x_p\})/\mathbb{Z}$

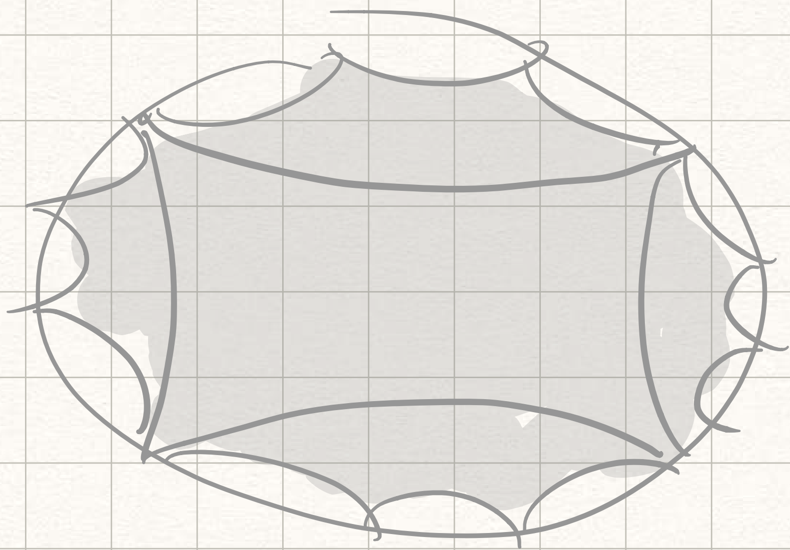
ex:





$\partial(\mathbb{F}_2, \emptyset) = e.$

ex:



$\partial(\langle a,b\rangle, [a,b]) = S^1$.

ex: $S^3 \setminus (\text{hyp knot} = K)$

$S^3 \setminus K$ admits a hyp struct w/ cusps, $\pi_1(S^3 \setminus K)$ acts by isometries on $\mathbb{H}^3$.

$\partial(\pi_1(S^3 \setminus K), \mathbb{Z} \oplus \mathbb{Z}) = S^2$.

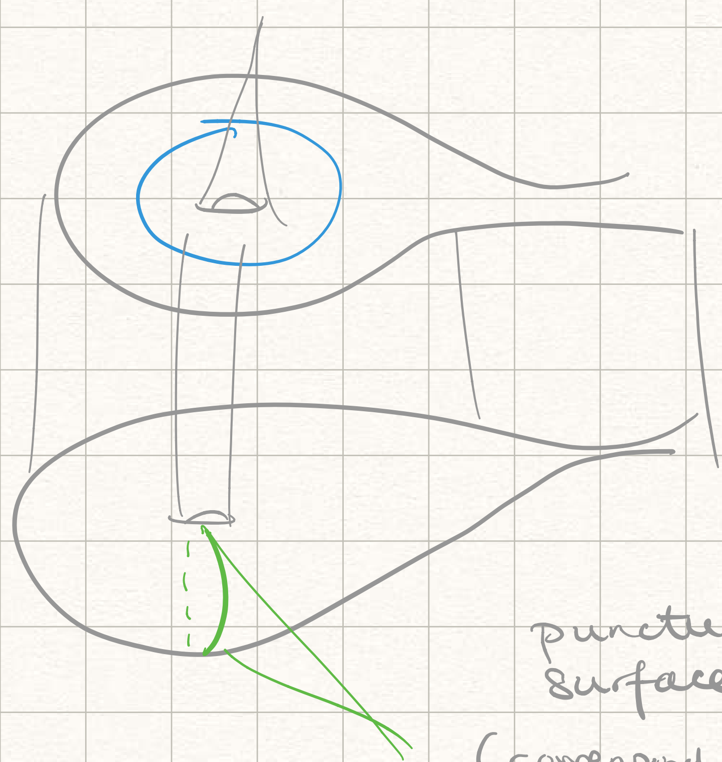
Collection of subgrps, $\{H_1, H_2, \dots, H_n\}$ of a (hyp) grp is <u>almost</u> malnormal if $H_i < G$, $H_i \cap g H_j g^{-1}$ is finite unless $i = j$ & $g \in H_i$.

Bowditch: if G hyp, $\mathcal{P}$ is an almost malnormal collection of q-convex subgrps consisting of finitely many conj. classes, then $(G, \mathcal{P})$ is rel. hyp. grp. grp. pair.

Tran + others: If G is hyp & $(G, \mathcal{P})$ is a rel. hyp. pair,

$$\partial(G, \mathcal{P}) = \partial G, \quad \partial P \sim pt \ \forall\, p \in \mathcal{P}.$$

Hyp. cusped 3-manifold ∂ embeds in S^2.

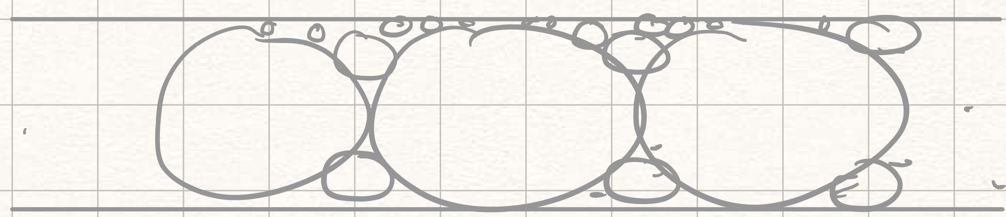


punctured surface $\times I$

(correspond to rigid piece)

$\partial(\pi_2, \{\langle a\rangle, \langle b\rangle, \langle c\rangle\})$

$=$



<u>Conj</u> (Generalization of conj of Cannon, Kapovich & Kleiner).

If a non-elmentary rel. hyp. grp pair (G, P) has planar $\partial(G, P)$ that does not have cut pts, then G is virt. isom to Kleinian grp.