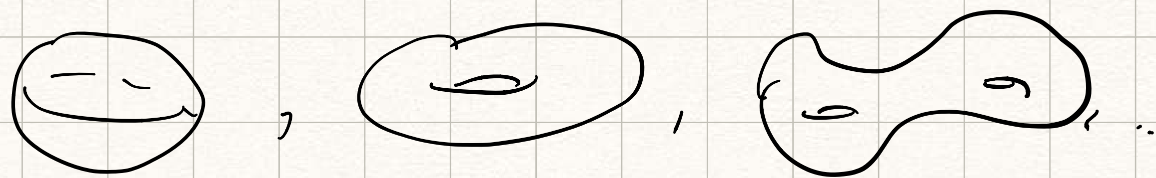
Hyperbolic Structures on Surfaces

Surface (2-dim, compact manifs w/o boundary).

There's a classification.



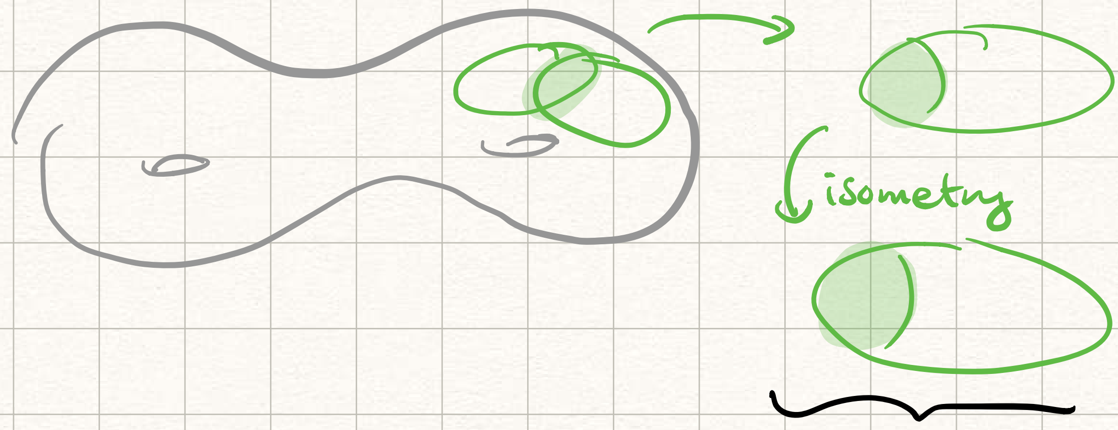
classification by genus.

Uniformization thm

For each Σ_g, there's a unique metric of ct curvature.

For every surface of $g \geq 2$, carries a hyperbolic structure, ie, a Riemannian metric of ct curvature -1,

or equiv, an atlas



hyperbolic plane.

Q: 1) How does the hyp. plane look like?

2) How do we endow a surface w/ a hyp. struct?

3) How does the space of all possible hyp. structs look like?

4) Beyond hyp..

re-listen (models of hyp).

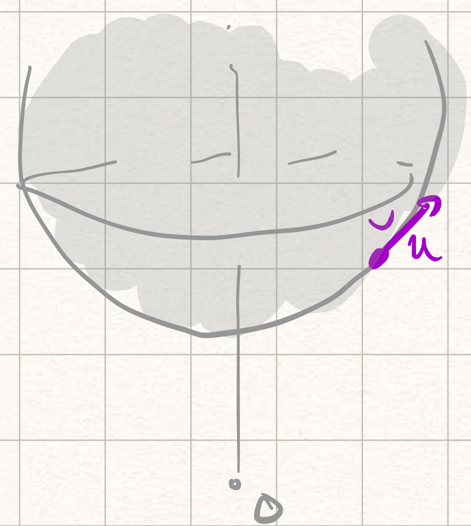
1) Hyperbolic Plane

Hyperboloid model

$(\mathbb{R}^3 = \mathbb{R}^{1,2}, \langle , \rangle_M)$

$\langle v, w \rangle = - v_0 w_0 + v_1 w_1 + v_2 w_2$

$$\mathcal{H} = \{ v \in \mathbb{R}^{1,2} \mid \langle v, v \rangle_M = -1, \; v_0 > 0 \}$$



$T_v \mathcal{H} = v^{\perp, \langle , \rangle_M}$ orthogonal complement of v, wrt $\langle , \rangle_M$.

metric in $\mathcal{H}$ → $(g_v^{\mathcal{H}}) = \langle , \rangle_M \big|_{v^\perp}$

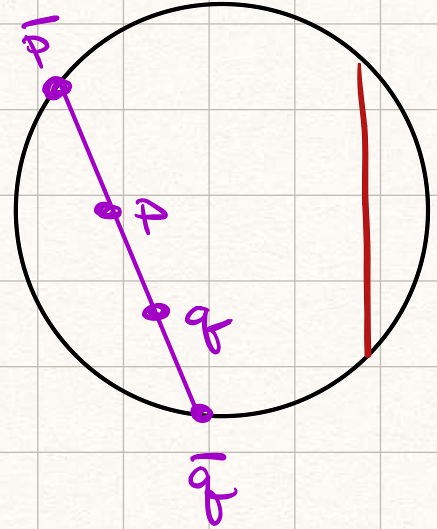
geodesics = intersections of $\mathcal{H}$ w/ planes through O.

Klein model:

$$\mathbb{B} = \{ \ell = \mathbb{R}v \in P(\mathbb{R}^{1,2}) \mid q|_\ell < 0 \}$$

where $q(v) = \langle v, v \rangle_M$.

geods = straight lines



distance fn:

$$d(p,q) = |\log|cr(\bar{p}, p, q, \bar{q})||.$$

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then the **cross-ratio** is def'ned as:

$$cr(z_1, z_2, z_3, z_4) := \frac{z_1 - z_2}{z_1 - z_3} \frac{z_4 - z_3}{z_4 - z_2}$$

Mobius transfo is a map
$f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C},$$

st $ad - bc = 1$.

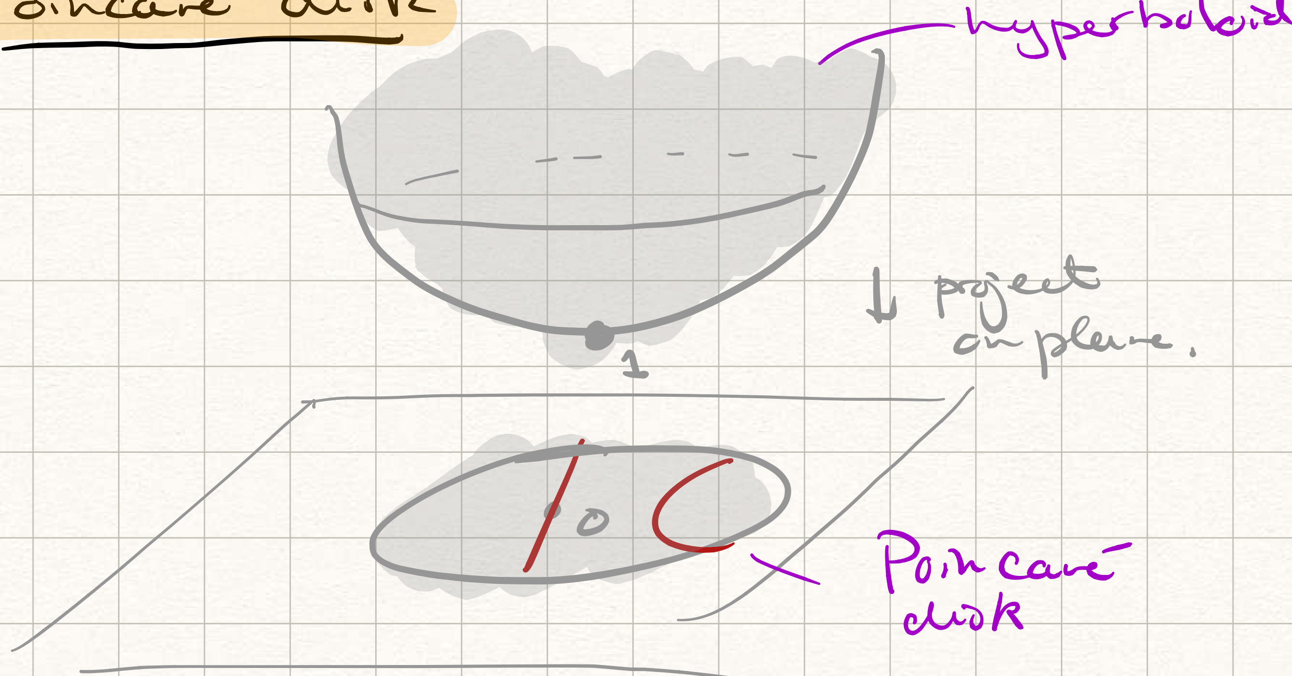
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

in $\mathbb{P}$

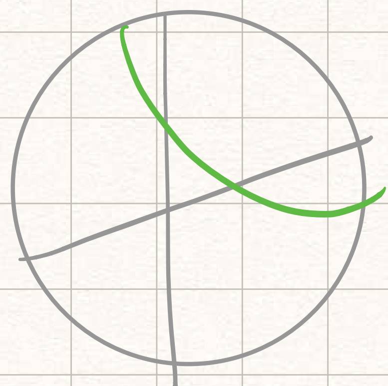
So $\text{Mob}(\hat{\mathbb{C}}) = PSL(2, \mathbb{C})$.

Fact: Mobius trans are conformal and maps circles to circles.

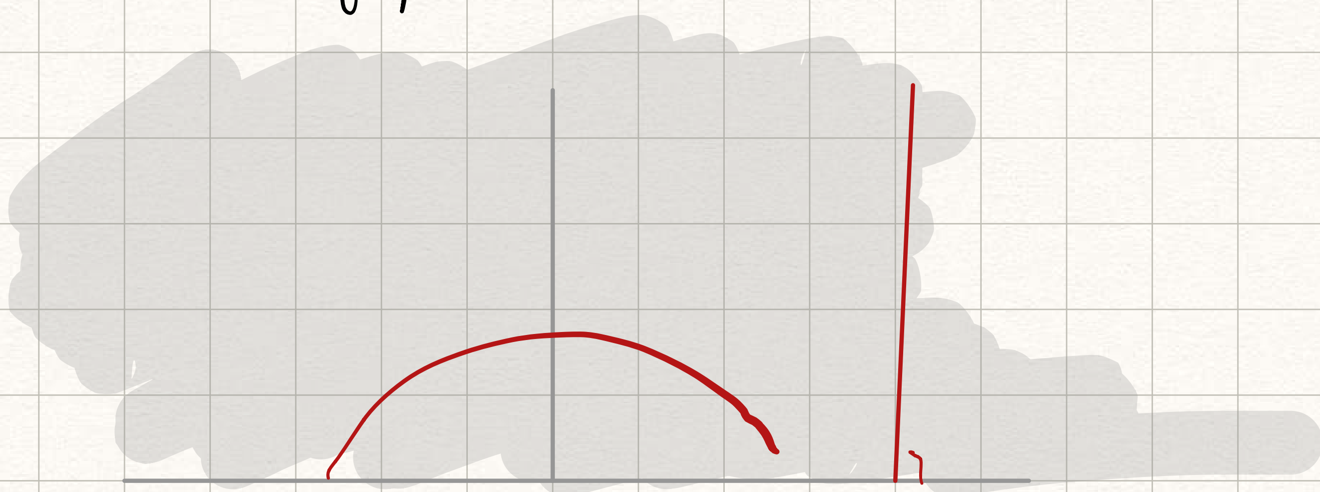
Poincaré disk



- geodesics are circ. arcs
- $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$
  the disk
- $g_z^{\mathbb{D}} = \dfrac{1}{(1-|z|^2)^2} \langle , \rangle_E$
- angles are the same,
  but sum of angles $< 180°$.



Upper half-plane model $\mathbb{H}$



- geods are also circs $\perp$ boundary.
  → to see this, can transform:

$\mathbb{D} \to \mathbb{H}$

$z \mapsto \dfrac{z-i}{z+i}$ ⇝ a Mobius trans.

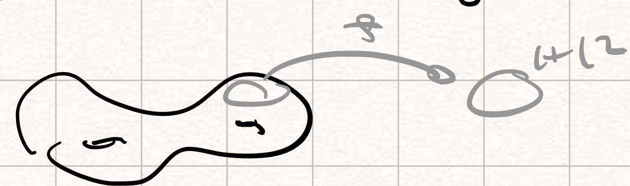
$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$

$g_z^H = \dfrac{1}{(\operatorname{Im} z)^2} \langle \, , \, \rangle_E$

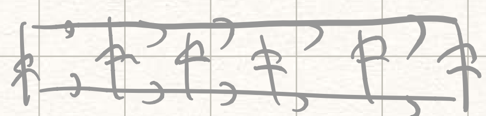
$\operatorname{Isom}(\mathbb{H}, g^H) = \{ f \in \operatorname{Mob}(\mathbb{C}) \mid f(\mathbb{H}) = \mathbb{H} \}$
$= PSL_2(\mathbb{R})$

2) Hyperbolic struct on S_g

→ charts



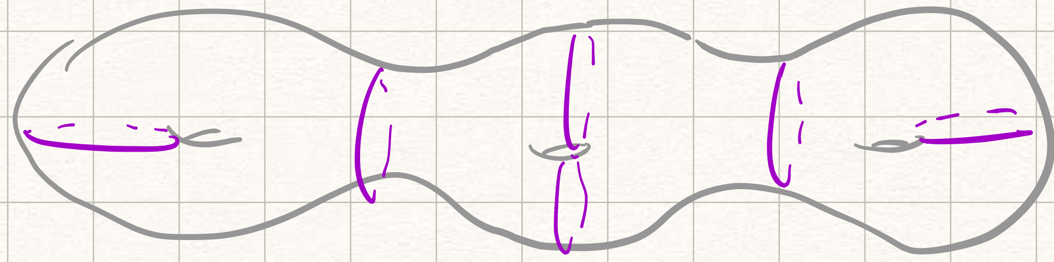
→ $\mathbb{R}^2$ + tesselation OR sym. grp Γ



$\Gamma < PSL_2(\mathbb{R})$, $\Gamma \cong \pi_1(S_g)$, where

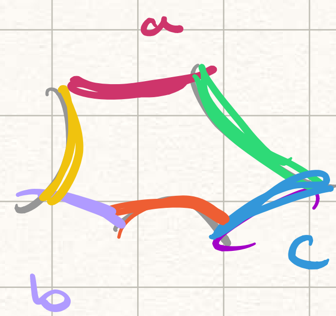
$\pi_1(S_g)$
$= \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle.$

→ Cut into simple pieces



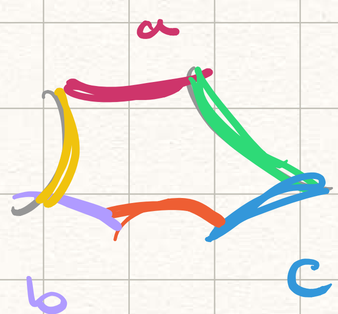
Take S_g into pairs of pants.

Take a pair:
& look at
struct.



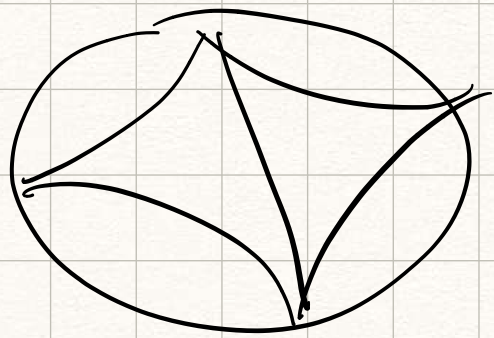
Here we have

- $3g-3$ ptwise disjoint geods scc.
- $2g-2$ pairs of pairs



there is a unique right-angled hexagon in $\mathbb{H}^2$ w/ these side-lens.

N.B: Also can do reverse:
given $l_1, l_2, l_3 \in \mathbb{R}_{>0}$, $\exists!$ hyperbolic pairs of pants w/ these side lengths.



that Δ.