What we really care about are marked hyp. structures.

Marked hyp. structures

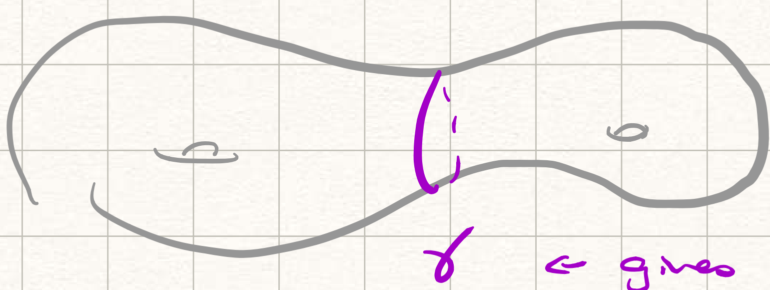
$S_g \xrightarrow{f} X$ marking

$$\mathrm{Hyp}(S_g) = \left\{ (X, f) \;\middle|\; \begin{array}{l} X \text{ hyp. surface} \\ f: S_g \to X \\ \text{homeo} \end{array} \right\} \Big/ \sim$$

where $\sim$ is by

$$\begin{tikzcd} S_g \arrow[r, "f"] \arrow[dr, "f'"'] & X \arrow[d, "\text{isom}"] \\ & X' \end{tikzcd}$$

So two marked space are equiv if there's an isom sending marking to marking.

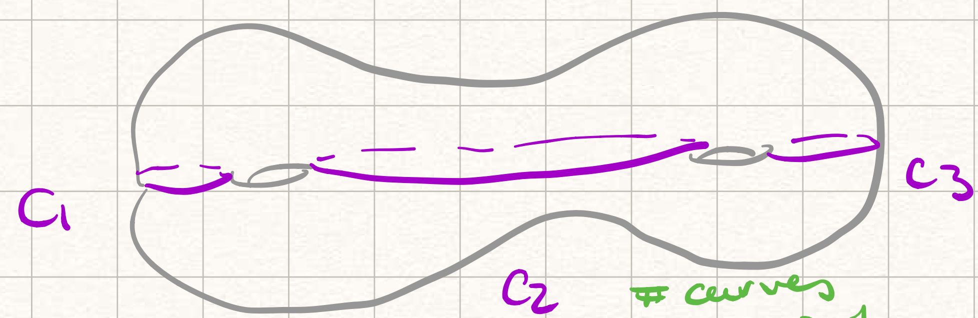


$\leftarrow$ gives us a way to name curve.

$$h \in MCG(S_g) = Homeo^+(S_g)/Homeo^+_0(S_g)$$
$$\curvearrowright \quad Hyp(S_g) \text{ by } \gamma \mapsto h\gamma$$

Fenchel - Nielsen parameters

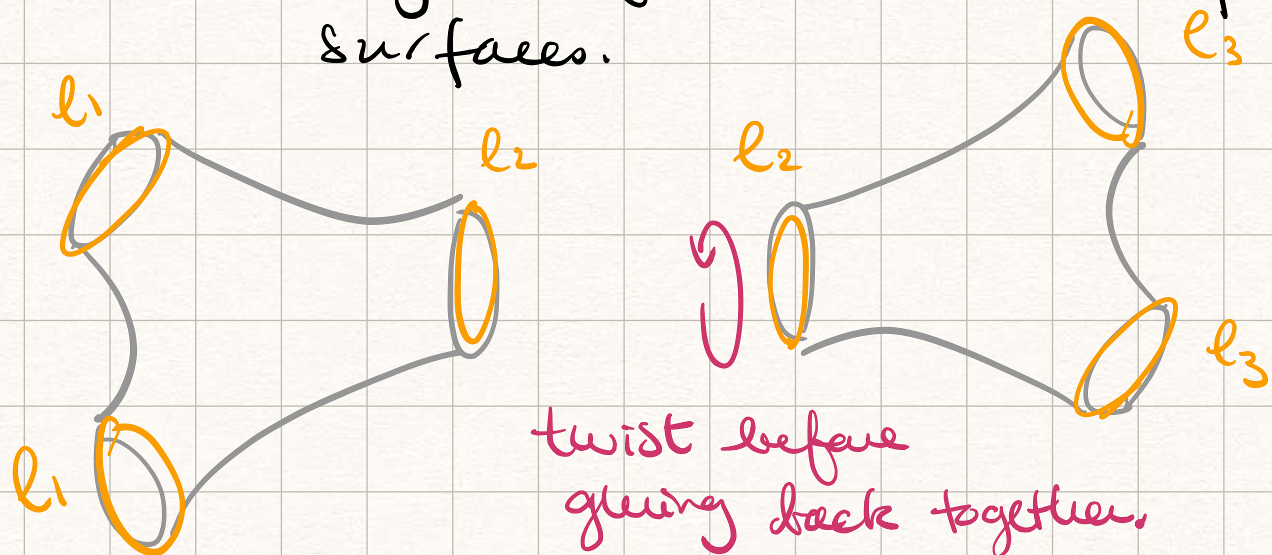
"Parametrize marked hyp struct using pair of pants coords."



\# curves pair of pants

$$(x, f) \to l_1 \;..\; .. \; l_{3g-3} \in \mathbb{R}_{>0}$$
$$\tau_1 \ldots \tau_{3g-3} \in \mathbb{R}$$

lengths of scc c_i decomp surfaces.



twist before gluing back together.

thus $\mathbb{R}^{2(3g-3)} \cong \mathrm{Hyp}(S_g)$

↳ usually write $\mathbb{R}^{6g-6}$

re-listen rmk.

... hard to write down how F-N coords. change.

<u>Wolforts formula</u>:

$$\sum_{i=1}^{3g-3} dl_i \wedge d\tau_i$$ wedge prod. is invariant under MCG action.

another rmk..

$$\mathrm{Hyp}(S_g) = \{(X = \mathbb{H}^2/\Gamma, f)\}/\sim$$

where $\Gamma < PSL_2(\mathbb{R})$.

(?) re-listen to follow

where $\sim$:

This marking induces map on π_1 to $PSL_2(\mathbb{R})$.

$$f_* : \pi_1(S_g) \longrightarrow \Gamma < PSL_2(\mathbb{R})$$

So get a map:

$$\mathrm{Hyp}(S_g) \xrightarrow{\text{holonomy}} \mathrm{Hom}(\pi_1(S_g), PSL(2,\mathbb{R}))/PGL(2,\mathbb{R})$$

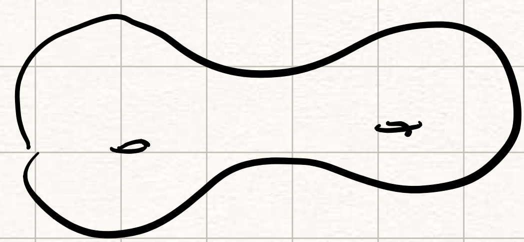
This map is injective, think of

$$\mathrm{Hyp}(S_g) \subset \mathrm{Hom}(\pi_1(S_g), PSL(2,\mathbb{R}))$$
$$\Psi$$
$$\rho$$

Note: Then ρ has discrete image and is injective.

$$\rho(\pi_1(S_g)) < PSL(2,\mathbb{R})$$

Q: How can you recover Hyp(S_g) from Hom(π_1) ?



$$\pi_1(S_g) = \langle a_1, b_1, \dots, a_g b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle .$$

Given $\rho \in \mathrm{Hom}(\pi_1(S_g), PSL(2,\mathbb{R}))$

→ so $\mathbb{Z}$ embeds.

same hom type as line

$$\begin{tikzcd} & \mathbb{Z} \arrow[d] \\ & \widetilde{PSL(2,\mathbb{R})} \arrow[d] \\ \pi_1(S_g) \arrow[ur, "\text{? can I do } \tilde{\rho}"] \arrow[r, "\rho"] & PSL(2,\mathbb{R}) < \mathrm{Homeo}^+(S^1) \\ a_i \arrow[r, mapsto] & A_i \end{tikzcd}$$

same hom. top type as S^1

Euler number

$eu(\rho)$

$= \mathrm{lift}\left(\prod [\tilde{A}_i, \tilde{B}_i] = \prod [\tilde{\rho}(a_i), \tilde{\rho}(b_i)]\right)$

w/ one pt to zero.

$\in \mathbb{Z} \cap [2-2g, 2g-2]$

Thm (Goldman '89–92)

The connected components of
$\mathrm{Hom}(\pi_1(S_g), PSL(2,\mathbb{R}))/PGL(2,\mathbb{R})$
are distinguished by
Euler number (eu), and ρ is
faithful w/ discrete image
$\Leftrightarrow$ $|eu(\rho)| = 2g-2$,

ie: $eu^{-1}(2g-2) = \mathrm{Hyp}(S_g)$

There are other grps, ie, $Sp(2n,\mathbb{R})$
which have the same features.

$$\begin{tikzcd} \mathbb{Z} \arrow[d] \\ \widetilde{Sp(2n,\mathbb{R})} \arrow[d] \\ Sp(2n,\mathbb{R}) \end{tikzcd} \quad \rightsquigarrow \quad \text{generalizations of Euler number}$$

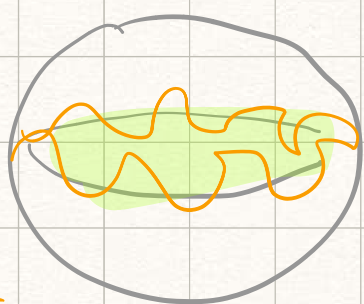
$\rightsquigarrow$ maximal representation.

Quasi-Fushian reps

$(X,f) \in Hyp(S_g)$

$\rho = f_*$
$\in Hom(\pi_1(S_g), PSL_2(\mathbb{R}))$

$\rho : \pi_1(S_g) \to PSL_2(\mathbb{R})$
$< PSL(2,\mathbb{C})$
$= Isom^+(\mathbb{H}^3)$



preserve fractal circ of boundary.

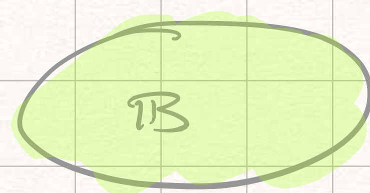
→ these are called quasi-Fushian reps.

Convex projective reps

$\rho : \pi_1(S_g) \to PSL_2(\mathbb{R}) < PSL_3(\mathbb{R})$

$PSL_2(\mathbb{R}) \cong SO^\circ(1,2)$

$PSL_3(\mathbb{R}) \curvearrowright \mathbb{R}P^2$



Klein model

deform

Strictly convex

this is C^1 but nowhere C^2.

Hitchin component for $SL_3(\mathbb{R})$.