Lem: $N(V) \to X$ (npccc) is a local isom

carrier

immersed hyperplane

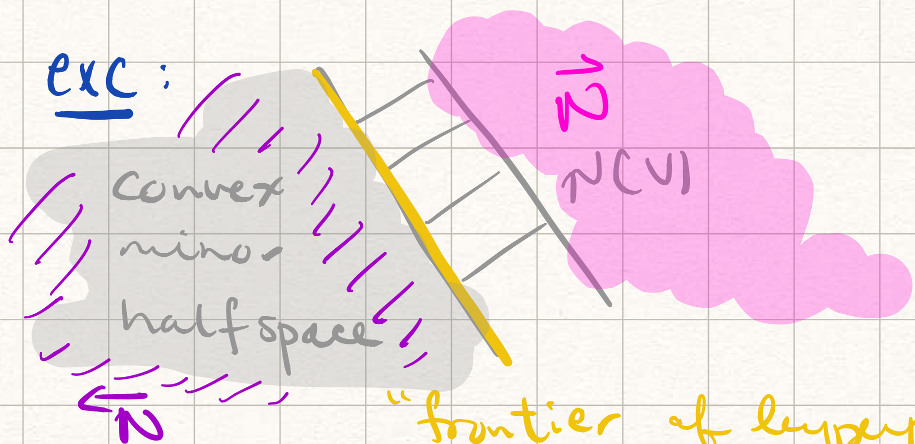
Cor: $\widetilde{N(V)} = N(\tilde{V})$

In particular, if X simply connected, carriers of $\tilde{X}$ are convex subcomplexes.

CAT(0)

(The hyperplanes don't crash into themselves due to local isoms).

exc:



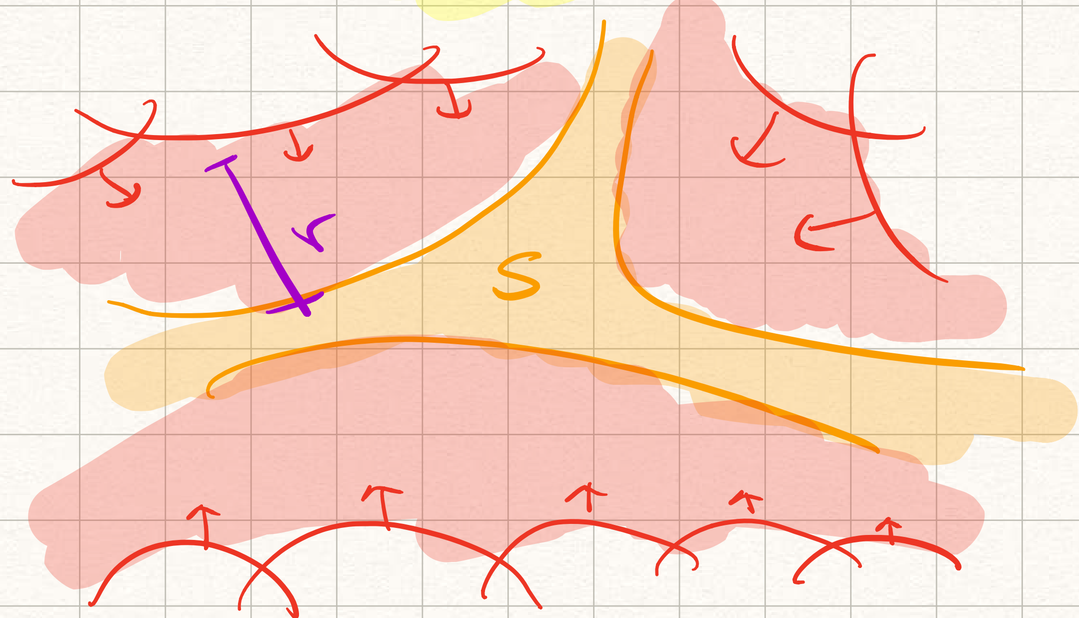
"frontier of hyperplane"

→ convex inside $N(V)$

$\tilde{Y} \subset \tilde{X}$ is convex $\iff$ $\tilde{Y}$ is the intersect of minor halfspace.

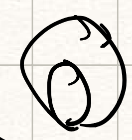
Let $\tilde{X}$ be a δ-hyp. CAT(0) cc.
Let $S \subset \tilde{X}$ be a κ-qconvex subspace.
Then $\exists r$ st

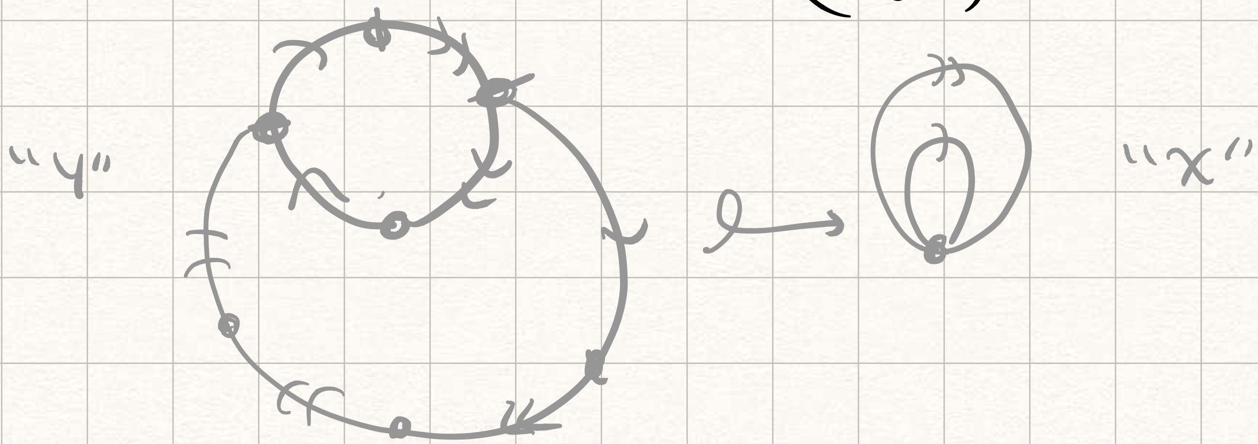
$$\mathrm{hull}(S) = \bigcap_{S \subset \vec{N}} \vec{N} \subset N_r(S)$$



hull is r away from S.

Cor: Let X be a compact npccc, w/ $\pi_1 X$ hyp. Let $H \subset \pi_1 X$ be q-convex subgrp w/ Y compact.
Then $\exists Y \underset{\text{local isom.}}{\looparrowright} X$, $\pi_1 Y = H$.

this generalizes our understanding of subgrps of $\pi_1($  $)$.



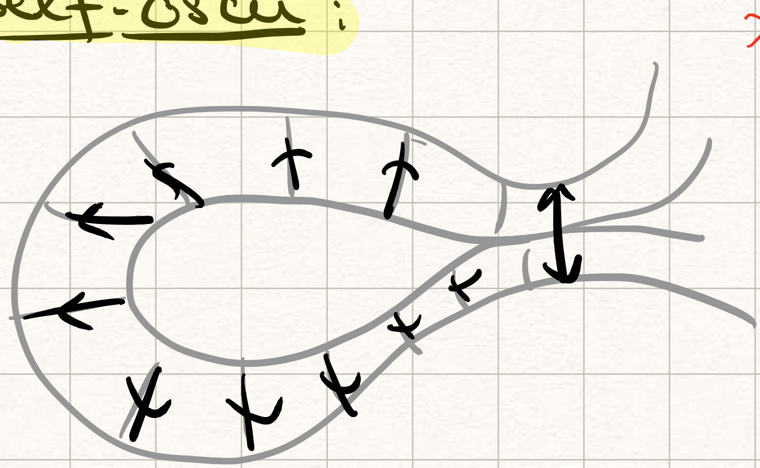
Special cube complexes

Npcc X is special if each immersed hyperplane embedding is:

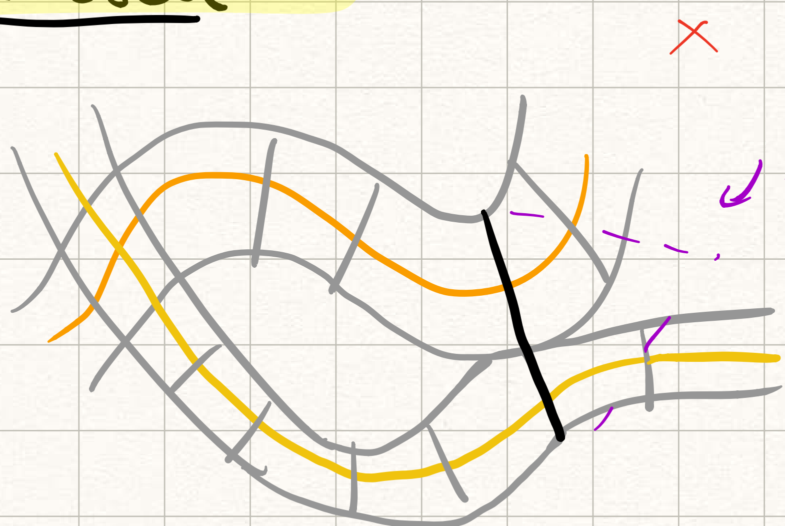
- two-sided
- doesn't self-osculate

& • no two hyperplanes interosculate.

self-osc :



interosc



ex: If $X \subset A \times B$, A, B graphs
then X special.

ex: Salvetti-complex.

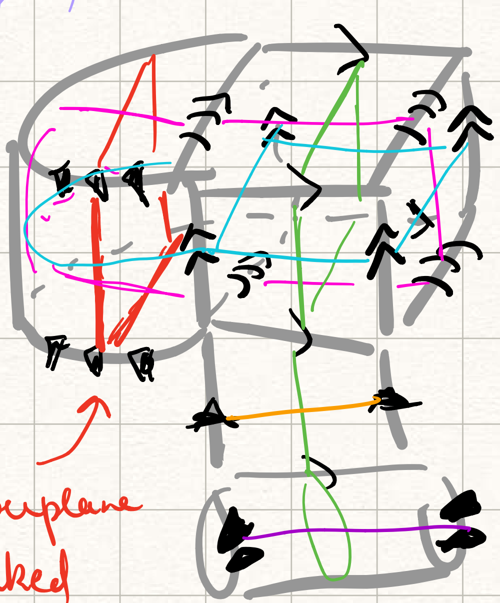
ex: CAT(0) cc is special.

! **Thm:** X is special
$\iff \exists$ local isom $X \to R$
where $R = R(\Gamma)$ is
the Salvetti complex.

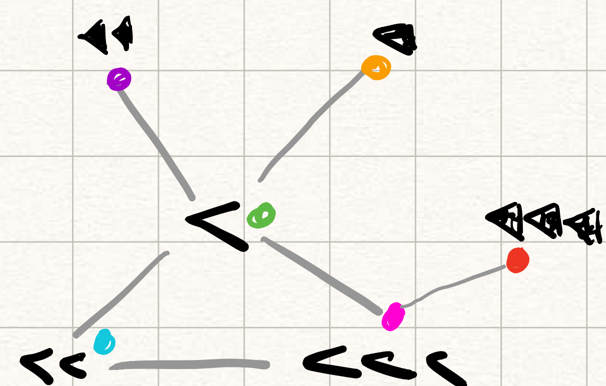
$(\Leftarrow)$ Pathology to pathology.
$(\Rightarrow)$ Use Γ crossing graph.

↳ Γ has a vertex for each hyperplane of X and an edge joins vertices where hyp. cross.

(☆)



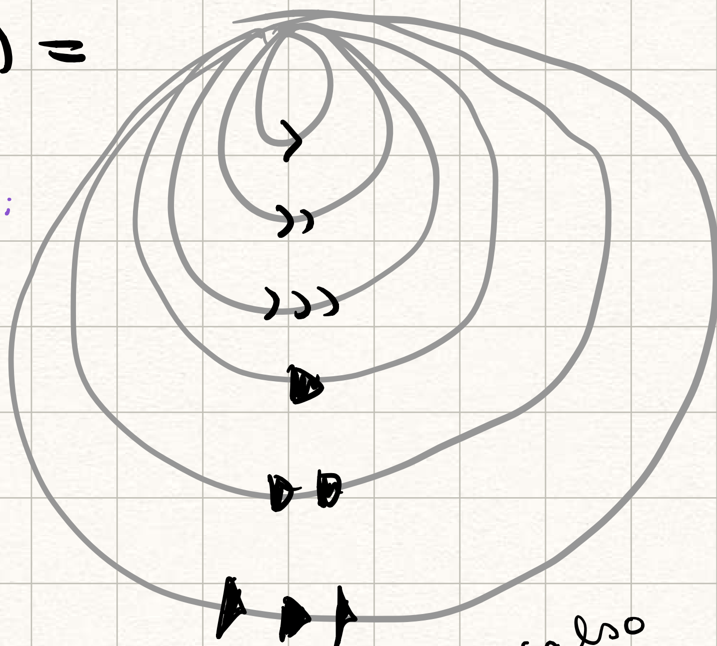
hyperplane marked by dual edges



$X \longrightarrow \Gamma$

(special)

$R(\Gamma) =$



so just map X onto this:

☆ : every sq maps to a sq in Salvetti cmplx! b/c sq = hyperplanes crossing.

(also ⇒ no missing sq)

self-cross, self-osc.

Used two-sidedness to say we can orient things consistently. (in the gluing).

Cor: If X is special, then $\pi_1 X \hookrightarrow$ RAAG!

So special cc form a bridge from:

Combinatorial geom. world $\longrightarrow$ comb. grp theory.

Virtual specialness ?

Thm (Agol) If X is compact & $\pi_1 X$ hyperbolic, then X is virtually special.

NB: Property T disj w/ cubulated.

Canonical Completion & Retraction

Thm: Let $Y \hookrightarrow X$ be a local isom w/ Y compact & X special.
There exists a finite cover $C(Y \to X) \to X$ st Y lifts to C. Moreover, $\exists$ retraction $C(Y \to X) \to Y$.

$$\begin{tikzcd}
 & C(Y \to X) \arrow[dl, shift left] \arrow[d] \\
Y \arrow[ur, hook, shift left] \arrow[r, hook] & X
\end{tikzcd}$$

Lem: RAAG is residually finite.

Lem: A retract of a res. fin. grp is **separable**.

re-listen.

$H \subset G$ separable if $H =$ intersection of fi subgrps of G.

Cor: $\pi_1 Y$ is sep in $\pi_1 G$ and hence in X.

Cor: If X is special and $\pi_1 X$ hyp and $H \subset \pi_1 X$ is q-convex, then H is sep.

③ Canonical retraction.

x ?