A **wallspace** is a set S w/ a collection $\mathcal{W}$ of walls. Each **wall** $w \in \mathcal{W}$ is a partition $S = \underset{p}{\overleftarrow{w}} \sqcup \underset{q}{\overrightarrow{w}}$.

Require $\#(p,q) < \infty$ for $p,q \in S$, where $\#(p,q)$ = number of wall separating p & q.

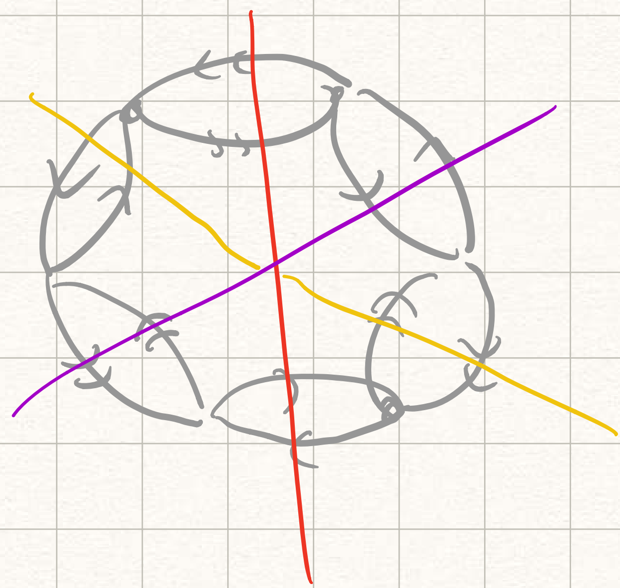
Walls:

ex: $\tilde{X}^0$ the vertices of CAT(0) cc

$\mathcal{W} \leftrightarrow$ hyperplanes.

ex: Coxeter grp:

walls $\longleftrightarrow$ reflection walls



$(S)=6$

what is this?

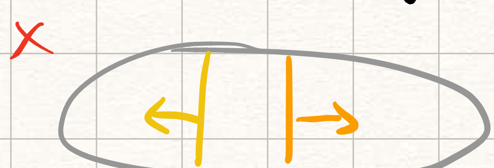
Dual cube complex

(Sageev's construction)

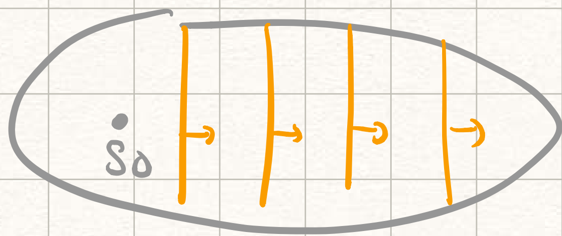
Construction takes as input a wallspace (S, W) and outputs CAT(0)-cc, $C(S,W)$.

0-cube: a choice of halfspace for each wall in W st

① no two choices are disjt.

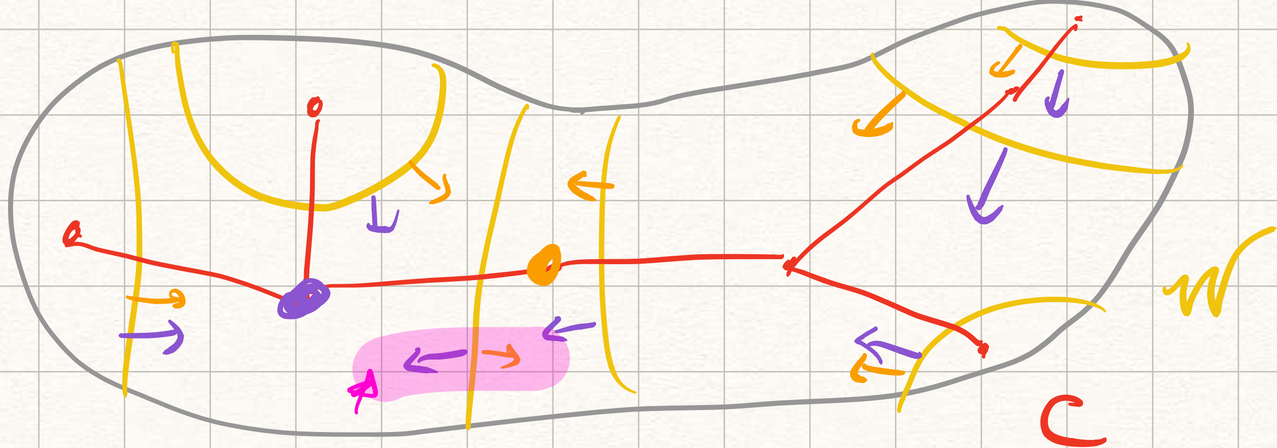


② all but finitely many choices contain some fixed pt $s_0 \in S$



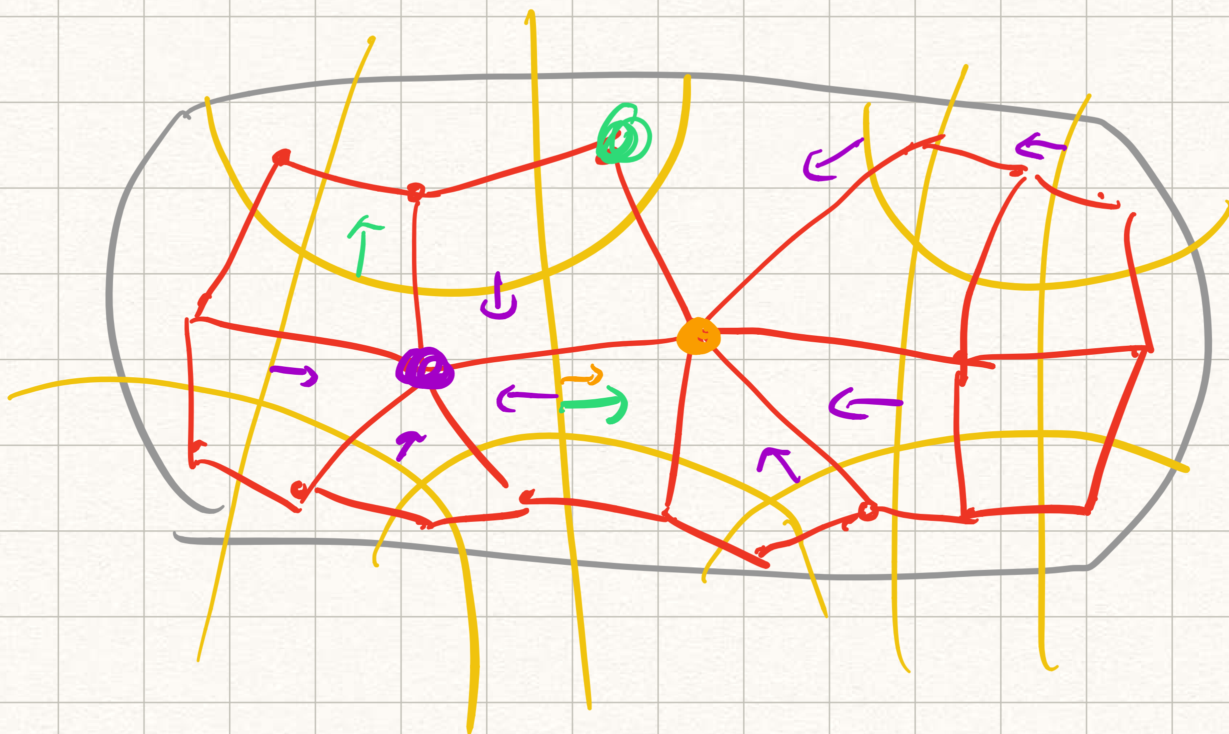
1-cube : join two 0-cubes if they differ exactly on one wall.

then add an n-cube whenever its (n-1)-skeleton is there.

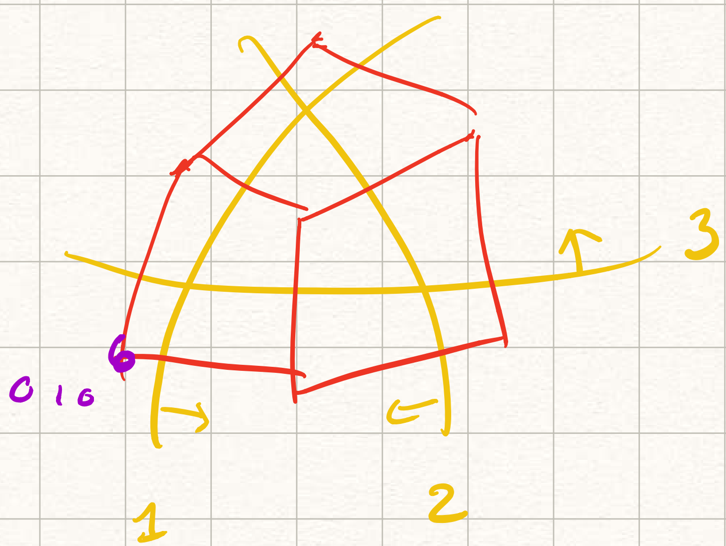


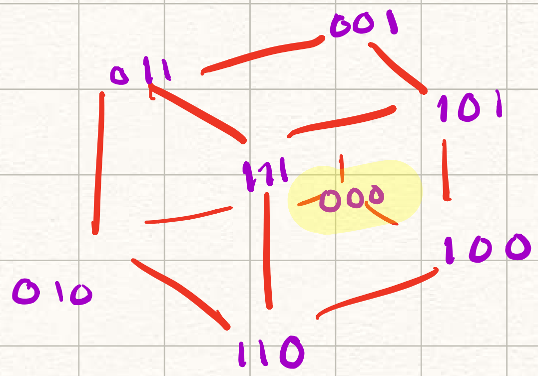
Note: only finitely many don't contain •.

*: the 1-cube b/w •, • conveys the one halfspace flip.



flip → get 0-cube



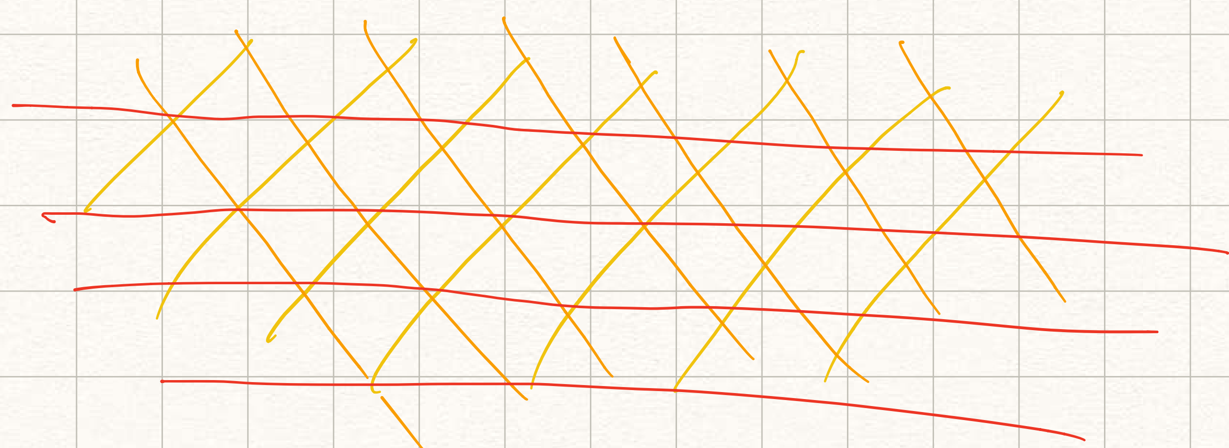


000 not a "canonical 0-cube":

0-cubes exist in dual, just orient all walls towards a pt $s \in S$, obtain canonical 0-cube C_s

exc: $G(S, W)$ is npc & simply connect'd.

exc: Compute the dual to a system of n families of parallel walls in $\mathbb{R}^2$.

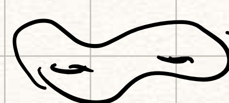


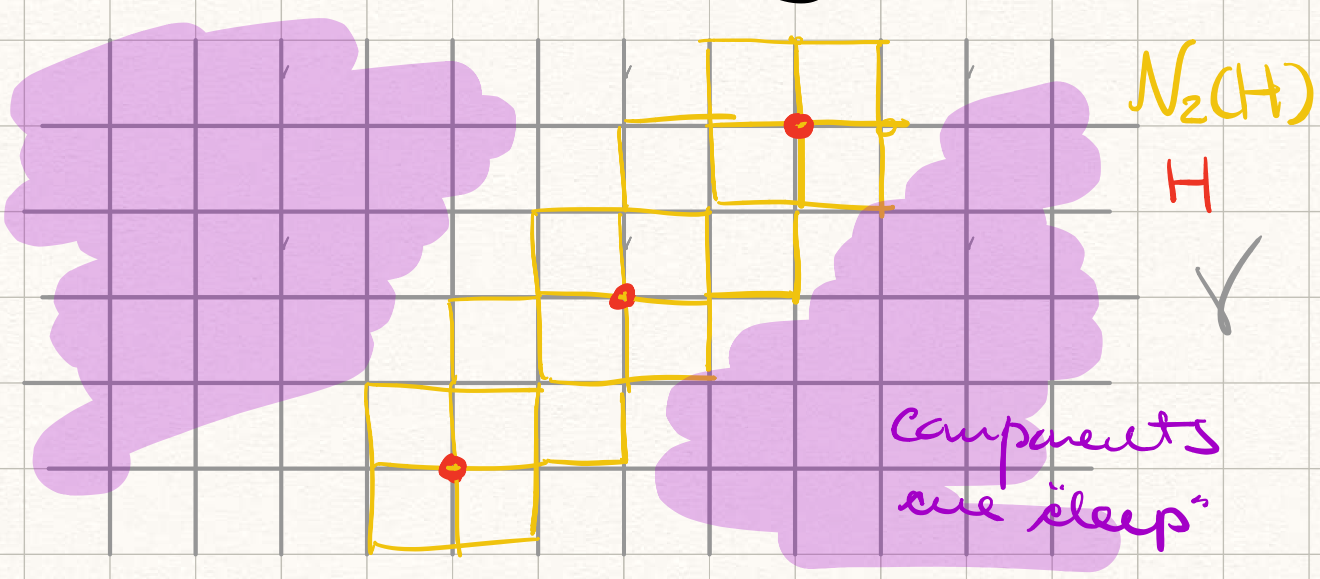
exc: W does not have three pairwise crossing walls
$\Leftrightarrow$ every 0-cube of $C(W)$ is canonical.

Let G be fg grp w/ Cayley graph Γ.

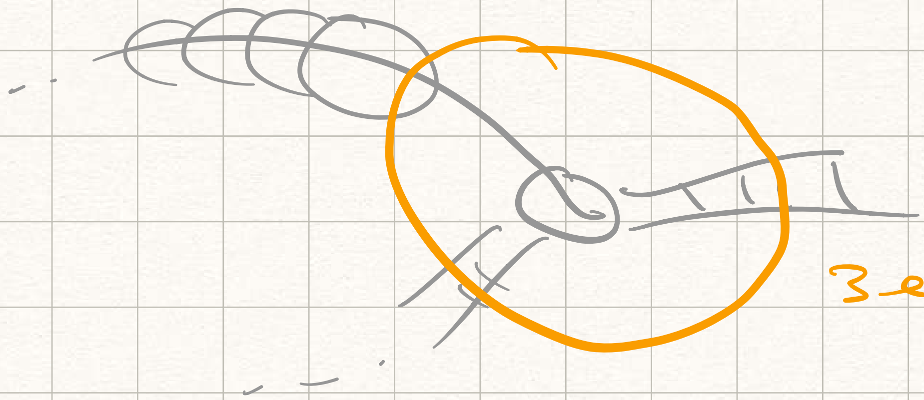
A subgrp $H \subset G$ is co-dim-1 if: $\exists r > 0$, st $\gamma - N_r(H)$ has at least two H-orbits of components K that are deep in the sense that $K \not\subset N_s(H)$ for any $s > 0$.

ex: $\mathbb{Z}^n \subset \mathbb{Z}^{n+1}$ is co-dim 1

any $\mathbb{Z} \subset \pi_1($  $)$ co-dim 1.

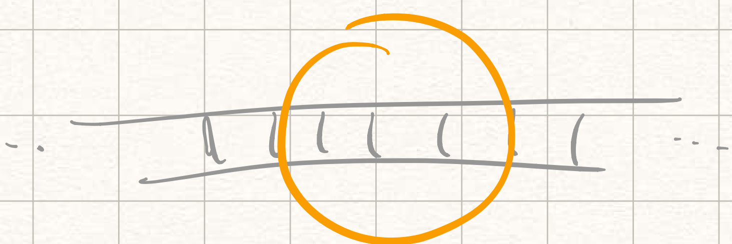


Lem : $H \subset G$ is co-dim 1 $\iff$ $H \backslash \Gamma$ has more than one end (as a Schreier graph).

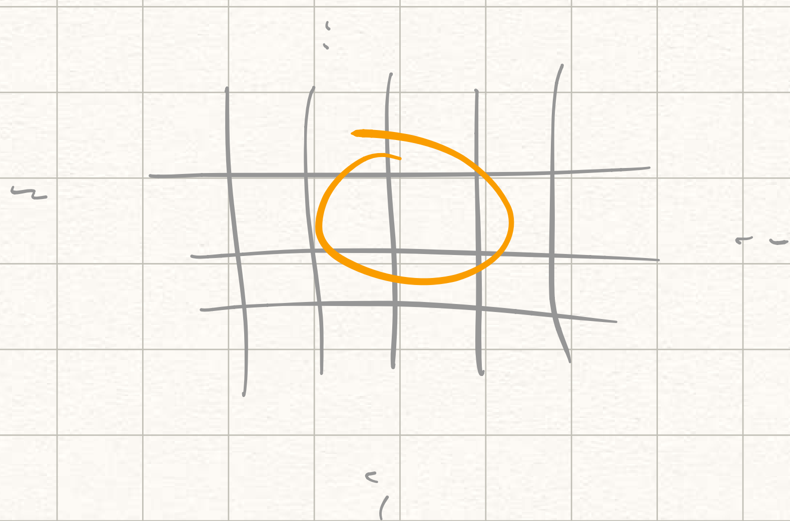


3 ends

Remove compact subgraph & get 3 infinite "deep" component.



2 ends



1 end.

$H \backslash \gamma$ = diag remove H compact.

missed something.

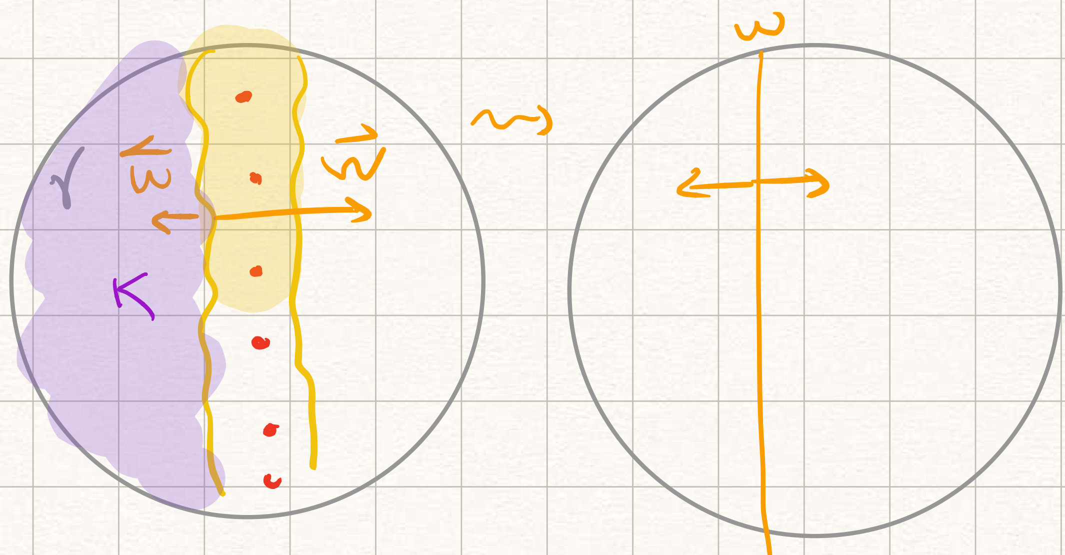
Co-dim-1 subgrp $\Rightarrow$ Wallspace
($\Rightarrow$ CAT(0) cc)

Choose r st $\gamma - N_r(H)$ has deep component K.

Let $W = \{\overleftarrow{w}, \overrightarrow{w}\} = \{K, \gamma - K\}$.

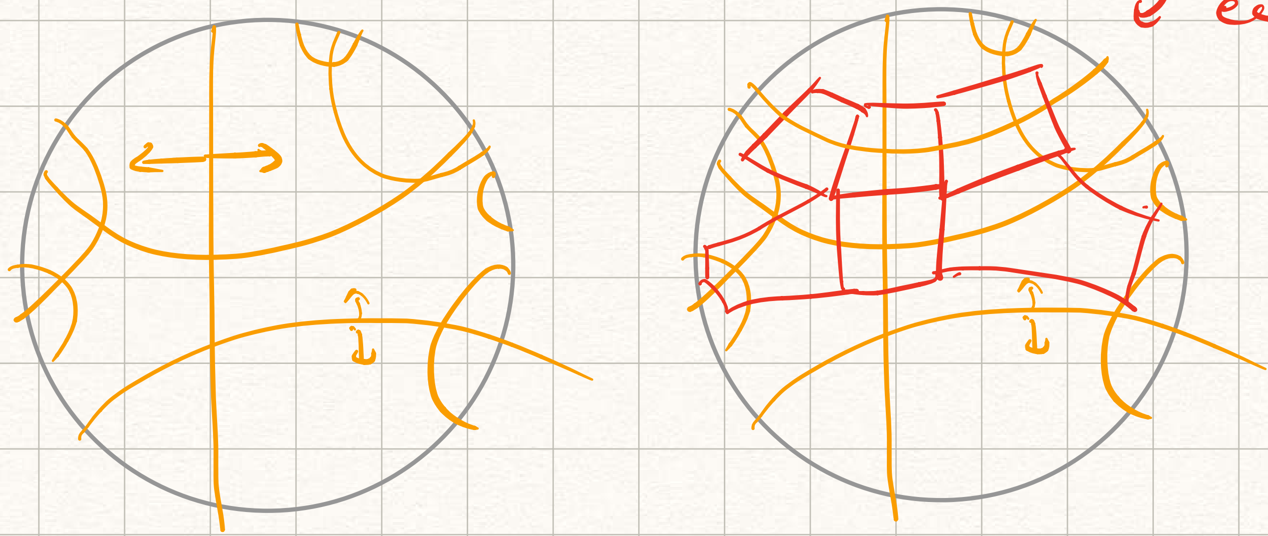
Let $\mathcal{W} = \{gW : g \in G\}$, $S = \gamma$.

This yields a wallspace w/ G-action $\Rightarrow$ obtain a dual w/ G action.



Now take G-translates.

take dual cc.



This construction works w/ several co-dim 1 subgrps.

exc: Why is $\#(p,q) < \infty$

Lem: Stabilizers of hyperplanes of $C(W)$ are commensurable w/ conjugates of H.