Statement of Research Interests

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1 Introduction

My long-term research ambition is to improve the algorithmic framework for computing with infinite groups.

During my PhD, I studied finitely generated left-orderable groups and formal languages. A group G is *left-orderable* if there exists a strict total order $\prec$ on the elements of G which is invariant under left-multiplication, $g \prec h \iff fg \prec fh, \quad \forall g, h, f \in G$. Equivalently, we often encode a left-order $\prec$ in terms its associated *positive cone* $P_\prec := \{g \in G \mid 1 \prec g\}$.

The central goal of my research projects was as follows. Construct or identify groups with a left-order $\prec$ for which it is "easy" to algorithmically decide for any two elements g, h whether $g \prec h$. The Word Problem is about algorithmically deducing whether $g = h$, or equivalently $g^{-1}h = 1$ where $g^{-1}h$ is encoded as a word w over the alphabet X. Analogously and using the left-invariance of $\prec$, my research problem is about algorithmically deducing whether $1 \prec g^{-1}h$ by deciding whether w evaluates to an element of $P_\prec$. Furthermore, the search space is narrowed down by requiring that if $\bar{w} \in P$, then w is given in a particular "form", or more precisely, is accepted in a *formal language*. In that sense, we say that a left-order is of complexity C if its positive cone $P_\prec$ can be represented by a formal language L which evaluates to $P_\prec$.

A formal language is a set of words over a finite alphabet which is accepted by an automaton of varying complexity. The basic families of formal languages, in increasing order of complexity along the Chomsky hierarchy, are: *regular languages* (languages which are accepted by finite state automata), *context-free languages* (languages which are accepted by finite state automata equipped with a stack), context-sensitive languages (languages which are accepted by a linear bounded automaton), and recursively enumerable languages (languages which are accepted by a Turing machine). In addition to capturing some notion of computational difficulty, formal languages have the desirable property of being closed under *AFL operations* (union, concatenation, Kleene star, intersection with regular languages, homomorphisms and inverse homomorphisms).

In my research, I focused on positive cone languages of lower complexity, where C is either regular or *one-counter*. One-counter languages are a subclass of context-free languages requiring the accepting automaton to have only two stack symbols (including the empty stack symbol), whereas context-free languages are accepted by automata with arbitrarily many stack symbols. Informally it is "easy" to decide membership for a word in a regular language because it only requires finite memory. Similarly, a one-counter language only requires keeping track of one number.

I have completed two research projects in the last three years which has allowed me to fulfill my research goal of finding left-orders of low complexity for a number of groups, and have constructed tools for further progress. The first project "Formal Language Convexity in Left-Orderable Groups" [Su20] is published in the International Journal of Algebra and Computation (IJAC). The second project "Regular left-orders on groups" [ARS21] is a collaboration with Yago Antolín and Cristóbal Rivas, which is available on arXiv. The papers deal with various themes related to my research goal. I have presented my results along these themes below for clarity.

2 Research overview and results

2.1 Note on studying left-orderable groups with formal languages

When using the formal language framework to decide whether $g^{-1}h \succ 1$, the requirement that the word w representing $g^{-1}h$ is in some formal language evaluating to $P_\prec$ rather than allowing it to be in the entire pre-image of $P_\prec$ is a natural one. For the Word Problem, Anisimov's theorem states that groups for which the pre-image of the evaluation map of the identity are regular languages are of finite cardinality [Ani71]. Using Anisimov's theorem, we obtained an analogous conclusion for left-orderable groups.

Lemma 2.1 ([ARS21])**.** *A finitely generated group G has a regular pre-image left-order if and only if it is trivial.*

2.2 Closure properties of positive cones under formal languages

Studying positive cones through the lens of formal languages allows us to use AFL closure to our advantage, such as quite easily obtaining that the complexity of a positive cone language is independent of finite generating sets, and is stable under taking extensions of two groups with the same positive cone language complexity. I was able to show that regular positive cones are also closed under taking wreath products and finite index subgroups. Note that the stability of *finitely generated* positive cones under extension and finite index does *not* hold, making regularity a better property to study. For example, $\mathbb{Z}$ has only two finitely generated positive cones generated by 1 and -1 respectively, but $\mathbb{Z}^2$ does not have any finitely generated positive cones. Furthermore, $\mathbb{Z}^2$ is a finite index subgroup of the Klein bottle group $K_2 = \langle a, b \mid baba^{-1}$ which has finitely generated positive cone $\langle a, b\rangle^+$.

Theorem 2.2 ([ARS21])**.** *Let N and Q be finitely generated groups which admits regular positive cones. Then the wreath product $N \wr Q$ admits a regular positive cone.*

The intuition for the proof of Theorem 2.2 is quite elementary and illustrates the power of working with formal languages: construct a positive cone for $N \wr Q$ by writing their elements in a normal form $(q_1 n_1 q_1^{-1})(q_2 n_2 q_2^{-1}) \cdots (q_n n_m q_m^{-1})q$ with $q_1 \succ_Q \cdots \succ_Q q_m$, then choose a lexicographic order on the normal form. Use the structure of the normal form to construct a regular positive cone language for $N \wr Q$ using the regular positive cone languages of N and Q and the closure properties of AFL operations.

Theorem 2.3 ([Su20])**.** *Let G be a finitely generated group with a regular positive cone. If H is a finite index subgroup, then H also admits a regular positive cone.*

The intuition for the proof of Theorem 2.3 is as follows: every element of G is some finite distance far away from H and thus every path in the Cayley graph of G lies some finite distance away from H. If P is a positive cone for G which is regular, we can obtain a regular language for $H \cap P$ from the one from P. More formally, we call subsets having this property language-convex.

Definition 2.4 ([Su20])**.** Let L be a language over X. A subset $H \subseteq G$ is *language-convex with respect to* L if there exists an $R \geq 0$ such that for each $w \in L$ with $\pi(w) \in H$, the induced path p_w lies within distance R from H in the Cayley graph.

Proposition 2.5 ([Su20])**.** *Let X be a finite set which is closed under formal inversion, $X = X^{-1}$. Set $G = \langle X\rangle$. Let L be a regular language, and let $P = \pi(L)$ where π is the evaluation map onto G. Let H be a subgroup of G. If H is language-convex with respect to L, then there exists a regular language L_H such that $\pi(L_H) = H \cap P$.*

The generality of Proposition 2.5 can be used to prove a result about acylindrically hyperbolic groups. Calegari showed in 2003 that no fundamental group of a hyperbolic manifold has a regular geodesic positive cone [Cal03]. In 2017 Hermiller and Sunic showed that no free products admits a regular positive cone [HS17]. Our result is a generalization of both results.

Theorem 2.6 ([Su20])**.** *A quasi-geodesic positive cone language of a finitely generated acylindrically hyperbolic group cannot be regular.*

However, it leaves us with the question of whether we can remove the quasi-geodesic requirement.

Question 2.7. Can we find a regular (non-quasi-geodesic) positive cone for an acylindrically hyperbolic group?

Theorem 2.6 is a consequence of fact that every acylindrically hyperbolic group admits a hyperbolically embedded subgroup which is isomorphic to F_2 [DGO17], [Osi16].

Lemma 2.8 ([Su20])**.** *If H is a hyperbolically embedded subgroup of an acylindrically hyperbolic group G, then H is language-convex with respect to every quasi-geodesic language L.*

Therefore, if an acylindrically hyperbolic group were to have a regular quasi-geodesic positive cone, then F_2 would inherit such a positive cone contradicting [HS17].

2.3 Geometric interpretation of regularity

Theorem 2.6 is interesting from a geometrical perspective. A priori, having a regular positive cone seems like a purely computational property. However, as with the word problem, the complexity of a positive cone can reveal surprising information about its geometry.

Definition 2.9. A set is $P \subseteq G$ is *coarsely connected* if it is connected in the Cayley graph up to some R-neighbourhood, for $R \geq 0$.

Alonso, Antolín, Brum and Rivas in 2020 showed that if P is a subset with a regular language representation, then P is coarsely connected [AAR20]. Moreover, they showed that positive cones of non-abelian free groups are not coarsely connected and that while there are left-orderable hyperbolic groups with coarsely connected positive cones, these have to be very 'distorted' in the sense that for every quasi-geodesic parameter (λ, c) there are pairs of positive cone elements which cannot be joined by such a (λ, c)-quasi-geodesic [AAR20].

In this light, the result of Hermiller and Sunic that no free product admits a regular positive cone and Theorem 2.6 are weaker analogous statements on more general versions of free groups and hyperbolic groups respectively.

2.4 Language complexity is positive cone dependent

Whereas the complexity of a positive cone is stable under changing finite generating sets, extensions and taking language-convex subgroups, the complexity of a positive cone language is dependent on the positive cone in question.

We have found that certain solvable Baumslag-Solitar groups given by presentation

$$BS(1, q) = \langle a, b | aba^{-1} = b^q \rangle, \quad q > 1$$

have both regular and non-regular left-orders [ARS21]. This is a combination of two results. For the non-regular left-orders, we refer to the well-known semidirect product decomposition $BS(1, q) \cong \mathbb{Z}[1/q] \rtimes \mathbb{Z}$, where additionally $\mathbb{Z}[1/q]$ is well-known to not be finitely generated.

Lemma 2.10 ([ARS21]). *Suppose that G is an extension of $\mathbb{Z}$-by-N. If N is not finitely generated, then no lexicographic orders on G where $\mathbb{Z}$ leads is a regular left-order.*

Other left-orders $B(1, q)$ are by induced an embedding $\rho : B(1, q) \to \text{Homeo}^+(\mathbb{R})$ sending a to a map $x \mapsto qx$ and b to a map $x \mapsto x + 1$ [Riv10]. We have shown that some of these orders are regular.

Theorem 2.11 ([ARS21]). *The group $G = BS(1, q)$ has regular left-orders. Moreover, all regular-left-orders on G are induced by affine actions on $\mathbb{R}$.*

2.5 A new family of groups with finitely generated positive cones

Although most of my research focuses on formal languages and positive cones, finitely generated positive cones (which are a subset of regular positive cones) have the advantages of being easy to describe and of inducing an isolated point on the space of left-orders. Note that regular positive cones do not do this, for example $\mathbb{Z}^2$ has many regular positive cones but no isolated left-order. Unfortunately, not many examples of finitely generated positive cones are known. In his 2011 paper, Navas [Nav11] constructs an infinite family of groups given by the presentation $\Gamma_n = \langle a, b \mid ba^n ba^{-1} \rangle$ for $n \in \mathbb{Z}$, which have positive cones of rank 2. The author then poses the following problem: for every $k \geq 3$, find an infinite family of groups which admit a positive cone of rank k. I solved this problem completely by looking into finite-index subgroups of Γ_n.

Theorem 2.12 ([Su20]). *For every integer $m \geq 2$, and integer $n \geq 2$ of the form $n = m-1+mt$ for some odd integer t, there is a subgroup of index m in $\Gamma_n = \langle a, b \mid ba^n ba^{-1} \rangle$ which admits a positive cone of rank $m + 1$.*

The discovery of this particular pattern of groups was observed first in GAP by finding the rank of the abelianization of the finite index subgroups and using it as a lower bound for the rank of the non-abelianized version. Then, the statement was proven using a simple application of the Reidemeister-Schreier method.

A finitely generated positive cone for $F_2 \times \mathbb{Z}$ was found using the parameters $m = 6$, $n = 2$. Since these parameters do not fit the restriction of Theorem 2.12, the positive cone is not of rank 7. However, by following the steps of the proof we can observe that the rank is bounded by 7, and hence finite.

2.6 Crossing left-orderable groups with $\mathbb{Z}$

While the complexity of positive cones languages is well-behaved under many operations, combining groups themselves result in new left-orders which behave in surprising ways: ways which we have only begun to uncover. For example, something inherent about positive cones change when left-orderable groups are crossed with a group with as "uncomplicated" left-orders as $\mathbb{Z}$. Free products of left-orderable groups have no isolated left-orders [DNR14] and are known to be one-counter [DS20]. However, it was shown in 2018 that groups of the form $F_{2n} \times \mathbb{Z}$ have both isolated and non-isolated orders relative to their space of left-orders [MR18]. Since having a finitely generated positive cone implies having an isolated left-order, a natural question is whether $F_{2n} \times \mathbb{Z}$ has a finitely generated positive cone. I was able to answer this question in the affirmative for $n = 1$.

Theorem 2.13 ([Su20]). *There exists a positive cone for $F_2 \times \mathbb{Z}$ which is finitely generated as a semigroup.*

This of course leaves open the following question.

Question 2.14. Does $F_{2n} \times \mathbb{Z}$ for $n > 1$ have finitely generated positive cones?

Moreover, Theorem 2.13 was derived from the tools used to prove Theorem 2.12 by taking $F_2 \times \mathbb{Z}$ as a finite index subgroup of the braid group $\Gamma_2 = \langle a, b \mid ba^2ba^{-1} \rangle$. With the given presentation, Γ_2 has a finitely generated positive cone $P = \langle a, b \rangle^+$ which has a very natural geometry. Answering the following question as an intermediate step could be useful.

Question 2.15. What does the geometry of the finitely generated positive cone of $F_2 \times \mathbb{Z}$ look like under its standard generating set?

Crossing with $\mathbb{Z}$ does not only change topological properties of positive cones but computational ones as well. Again citing the result of Hermiller and Sunic [HS17], it is known that free products do not admit regular positive cones. A paper by Dicks and Sunic published in 2020 show that if a group G acts on a tree with trivial edge-stabilizer (such as free products), then it admit a quasi-morphism $\tau : G \to \mathbb{Z}$ which induce a left-order on G. The quasi-morphism has three properties which allows it to induce a left-order: $\tau(g) = 0 \iff g \in C$ for some left-orderable group C or C is trivial, $\tau(g^{-1}) = -\tau(g)$, and $|\tau(g) + \tau(h) - \tau(gh)| \leq 1$, and we call such a map an *ordering quasi-morphism*.

Interpreting the existence of τ under the lens of formal languages, we obtain that free products of left-orderable groups with regular positive cones admit a one-counter positive cone, which is the minimal admissible positive complexity for free products.

Proposition 2.16 ([ARS21]). *Let G be a group finitely generated by (X, π) and $\tau : G \to \mathbb{Z}$ an ordering quasi-morphism with kernel C. If there exists τ-transducer $\mathbb{T}$ and a regular language $\mathcal{C}_C$ such that $\pi(\mathcal{C}_C) = P_C$ is a positive cone for C, then $P_\tau \cup P_C$ is a one-counter positive cone, where $P_\tau = \{g \in G \mid \tau(g) > 0\}$.*

Moreover, this framework allows us to interpret crossing with $\mathbb{Z}$ as 'attaching a stack' to our group.

Proposition 2.17 ([Su20]). *Let G be a finitely generated by (X, π_X). Let C be a subgroup of G, and $\tau : G \to 2\mathbb{Z}+1 \cup \{0\}$ be an ordering quasi-morphism computable through a τ-transducer with kernel C. Let P_C be a positive cone for C, let*

$$\tilde{P} = \{(g, n) \in G \times \mathbb{Z} \mid \tau(g) + 2n > 0\} \cup \{(c, 0) \mid c \in P_C\}.$$

The set $\tilde{P}$ is a positive cone for $G \times \mathbb{Z}$.

Antolín, Dicks and Sunic were able to extend the ordering quasi-morphism to edge-groups which are relatively convex in their vertex groups [ADS18]. Further work on these quasi-morphisms could be in the following direction.

Question 2.18. Can we find ordering quasi-morphisms for amalgamated products? Can we construct one-counter positive cones from these quasi-morphisms?

Since RAAGs can be viewed as iterated amalgated products, we have the following related question.

Question 2.19. Can we find an ordering quasi-morphism for RAAGs and show that RAAGs have one-counter positive cones?

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