Computation in the Completion of the Free Group Algebra

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Overview

Goal

$\mathbb{Q}(F_n)$ can be embedded in a division ring D. Create an algorithm to do addition, multiplication and inversion in D.

Definitions

Group Ring: Let K be a field, and G a multiplicative group, finite or infinite. Then the *group ring* $K[G]$ is the set of formal linear combinations of the form

$$\sum_{g \in G} k_g g, \qquad k_g \in K$$

with finite support. Addition and multiplication are defined in the natural way as follows:

$$\sum_{g \in G} k_g g + \sum_{g \in G} l_g g = \sum_{g \in G} (k_g + l_g) g$$

$$\left(\sum_{g \in G} k_g g \right) \cdot \left(\sum_{h \in G} l_h h \right) = \sum_{g, h \in G} (k_g l_h) gh$$

Definitions

Division Ring: Is a nonzero ring in which every nonzero element has a multiplicative inverse.

N.B. Group rings are *never* division rings!

If the group is of finite order, it has zero divisors. If the group is not of finite order, then the inverse of $(1+g)$ is $1-g+g^2-\ldots$.

Embedding Theorem

Let $K[G]$ be the group ring of the ordered group G, and let D be the set of all formal sums $\sum_{g\in G} k_g g$ with *well-ordered support*. Then, with the natural addition and multiplication, D is a K-division algebra containing $K[G]$.

[Pas11]

ex:

$$(1+g)^{-1} = 1 - g + g^2 - \dots,$$

Here, we are interested in $K = \mathbb{Q}$, $G = F_2 = \langle x, y\rangle$.
Computation in this completion $D \supset \mathbb{Q}[F_2]$ is equivalent to computing L^2 invariants of certain nonpositively curved HNN extensions of a free group.

[BDHM15]

Problem

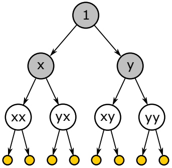
How do we make an algorithm that adds, multiplies and inverts infinite sums?

Insight 1

D is obtained by extending $\mathbb{Q}[F_2]$ to include inverses, where each element of $\mathbb{Q}[F_2]$ is a finite sum. Our hope is that we can encode the inverse of a finite sum with a "finite amount of data".

Insight 1

Restrictiong our problem to doing arithmetic on elements in D with support in the set of positive words T



We say that $A \in D$ has a **finite amount of data** $A = \sum_{\alpha \in T} a_\alpha \alpha$ satisfies the recursive relation on its coefficients

$$a_{\alpha\pi} = \sum_{\tau \in I} r(\pi, \tau) a_{\alpha\tau}, \qquad \pi \in J.$$

where I is the determining set of the recursive relation, J the set of children of I not in I, and $r(\pi, \tau)$ are the coefficients. [Zam15]

Intuition

Think of $1/3 = 0.33333\cdots = 0.\overline{3}$.

Rephrase Problem

- Find a minimal representation of elements in D with support in T.
- Do $+, \times, ^{-1}$ with such elements.

Key Insights 2

A has a finite amount of data if and only if $V_A := \{\partial_\alpha A : \alpha \in T\}$ if finite dimensional, and

- A basis for V_A corresponds to a minimal I for A.
- $\partial_\pi A = \sum_{\tau \in I} r(\pi, \tau)\partial_\tau A$.

$\partial_\gamma A := \sum_{\alpha \in T} a_{\alpha\gamma}\alpha$.

[Zam15], [FV92]

Given an element C, if we find a basis of V_C, and solve for the recurrence coefficients, $r(\pi, \tau)$ we win!

Algorithm

Let $C = A + B, \quad A, B, C \in D$. We input the finite determining set and recurrence coefficients of A, B, and expect the determining set and recurrence coefficients of C as an output.

1. Then $V_C \subseteq V_A + V_B$. Write $\partial_\alpha C = \partial_\alpha A + \partial_\alpha B \in V_A + V_B$, the indirect sum of the vector spaces generated by A and B. Add vectors $\partial_\alpha C$ in the table, with $\alpha \in T$ in level-order until we get an entire level of the tree T being linearly dependent. That set of α forms a determining set $\bar{I}$ for C.

*	$\partial_1 A$	$\partial_x A$	$\partial_y A$	$\partial_1 B$	$\partial_y B$
$\partial_1 C$	1	0	0	1	0
$\partial_x C$	0	1	0	2	0.5
$\partial_y C$	0	0	1	0	1

- Allows us to get a spanning set for V_C.
- Problem: $V_A \cap V_B \neq \emptyset$.

Algorithm

2. Realize each $\partial_\alpha C$ as an endomorphism $\partial_\alpha : V_C \to V_C$. $\partial_\alpha\big|_{V_C}$ is entirely defined by its action on the set $\{\partial_\tau C : \tau \in \bar{I}\}$. $I' \subseteq \bar{I}$ is the maximal complete subtree rooted at 1 with $\{\partial_\alpha\big|_{V_C} \in \mathrm{End}(V_C) : \alpha \in \bar{I}\}$ linearly independent.

$$\partial_1\big|_{V_C} = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & 1 & \dots & \dots \\ 1 & 2 & 0 & \dots & \dots \\ 0 & 0.5 & 1 & \dots & \dots \end{pmatrix}$$

Algorithm

3. Recover the $r(\pi,\tau)$ coefficients of C. using
$\mathbf{r}_\pi = [c_{ij}]^{-1}_{i,j\in\tau}\mathbf{c}_\pi, \qquad \mathbf{c}_\pi = (c_{\tau\pi})_{\tau\in I}, \qquad c_\alpha = a_\alpha + b_\alpha$ and

$$c_{\alpha\pi} = \sum_{\tau\in I} r(\pi,\tau)c_{\alpha\tau}, \qquad \pi \in J.$$

4. Find a maximally linearly independent set of $\{\partial_\alpha C : \alpha \in T\}$, where the $\partial_\alpha C$ are written with respect to spanning set of V_C $\{\partial_\tau C : \tau \in I'\}$

*	$\partial_1 C$	$\partial_x C$	$\partial_y C$	$\partial_{xx} C$
$\partial_1 C$	1	0	0	0
$\partial_x C$	0	1	0	0
$\partial_y C$	0	0	1	0

5. Repeat step 3.

Output

We have calculated the determining set for C and its recurrence coefficients, so we are done!

References

E Berkove, S. Dharia, T. Hsu, and R. Mcguigan, *The ℓ^2-Homology of Clean Complexes*, Preprint (2015).

M. Farber and P. Vogel, *The Cohn localization of the free group ring*, Mathematical Proceedings of the Cambridge Philosophical Society **111** (1992), 433.

Donald S. Passman, *The algebraic structure of group rings (dover books on mathematics)*, Dover Publications, 2011.

Olga Zamoruyeva, *Computation In A Localization Of The Free Group Algebra*, Master's thesis, San José State University, CA, USA, 2015.

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