# Introduction to $L^2$-Betti Numbers

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The goal of this presentation is to set up the framework for $L^2$ Betti Numbers from the point of view of von Neumann Algebras.

## Contents



# 1 Topological Group

We start with some basic definitions.

**Definition 1.** *A topology on $X$ is* discrete *if every subset of $X$ is open.*

**Definition 2.** *A* topological group $(G, \tau)$ *is a topological space $G$ (viewing $G$ as a set) equipped with the topology $\tau$, such that every $g \in G$ is also gives us a map $G \to G$ which is continuous with respect to $\tau$.*

# 2 Von Neumann Algebra

We will now define the space $\ell^2(G)$ we will be playing in:

**Definition 3.** *If $G$ is a discrete group, then view $G = \{g\}_{g\in G}$ as an orthonormal set, and $V_G$ the vector space over $\mathbb{C}$ generated by $G$. Then,*

$$\ell^2(G) := \{\sum_{g\in G} \lambda_g g \in V_G : \sum_{g\in G} |\lambda_g|^2 < \infty\}$$

Notice that by declaring $V_G$ a vector space over $\mathbb{C}$, we inherit an inner product

$$\langle \sum_{g\in G} \lambda_g g, \sum_{g\in G} \mu_g g\rangle := \sum_{g\in G} \lambda_g \bar{\mu}_g,$$

so $\ell^2(G)$ is a Hilbert space.

**Definition 4.** *Let $*_X, *_Y$ be the action of $G$ on $X, Y$ respectively. A map $f : X \to Y$ is said to be* equivariant *if*

$$f(g *_X x) = g *_Y f(x)$$

**Definition 5** (Group Von Neumann Algebra)**.** The Group Von Neumann Algebra $\mathcal{N}(G)$ *is the algebra of $G$-equivariant bounded linear operators $\mathcal{B}$ from $\ell^2(G) \to \ell^2(G)$. We denote it as*

$$\mathcal{N}(G) := \mathcal{B}(\ell^2(G))^G.$$

*where the $G$-action that is equivariant is understood to be left multiplication by $g \in G$.*

**Definition 6** (Von Neumann Trace)**.** *The* Von Neumann Trace *on $\mathcal{N}(G)$ is a map*

$$tr_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \qquad f \mapsto \langle f(e), e\rangle_{\ell^2(G)}.$$

Let us play with the simplest example:

**Exercise 1.** *Show that if $G$ is finite, show $\mathcal{N}(G) = \ell^2(G)$, and*

$$tr_{\mathcal{N}(G)}(\sum_{g\in G} \lambda_g g) = \lambda_e.$$

*Proof.* Since $G$ is finite, $\ell^2(G) = \mathbb{C}G$. We construct an isomorphism $\Phi : \ell^2(G) \to \mathcal{N}(G)$ in the following way: for $v, w \in \ell^2(G)$,

$$v \mapsto \Phi(v),$$
$$[\Phi(v)](w) = wv.$$

Since the product of two finite sums are finite, and $wv \in \ell^2(G)$, and $\Phi(v) : \ell^2(G) \to \ell^2(G)$ as wanted. $\Phi(v)$ is clearly $G$-equivariant, because for $g \in G$,

$$[\Phi(v)](gw) = gwv = g[\Phi(v)].$$

It is linear, as

$$[\Phi(v)](\mu_1 w_1 + \mu_2 w_2) = (\mu_1 w_1 + \mu_2 w_2)v = \mu_1 w_1 v + \mu_2 w_2 v = \mu_1[\Phi(v)](w_1) + \mu_2[\Phi(v)]w_2,$$

and bounded because

$$\begin{aligned}\|[\Phi(v)](w)\|_2 = \|wv\|_2 \leq \|w\|_2\|v\|_2 \\ \Rightarrow \|\Phi(v)\|_{\text{op}} \leq \|v\|_2 < \infty.\end{aligned}$$

Therefore, for all $v \in \ell^2(G)$, we have shown that $\Phi(v) \in \mathcal{N}(G)$. It remains to show that the map is an isomorphism. This map is a homomorphism because

$$[\Phi(vw)](u) = uvw = [\Phi(w)](uv) = [\Phi(v)]([\Phi(w)](u)),$$

and hence $\Phi(vw) = \Phi(v) \circ \Phi(w)$, where $\circ$ is the composition of functions.

The map is injective because if

$$[\Phi(v)](w) = 0 \qquad \forall w \in \ell^2(G),$$

it must be that $v = 0_{\mathbb{C}G}$. Now, suppose $M \in \mathcal{N}(G)$. Then, for some $v = \sum_{g\in G} \lambda_g g$,

$$M(w) = \sum_{g\in G} \lambda_g M(g) = \sum_{g\in G} \lambda_g g M(1_G) = wM(1_G)$$

Letting $v = M(1_G) \in \ell^2(G)$, we have

$$[\phi(v)](w) = M(w) \qquad \forall w \in \ell^2(G).$$

Therefore, under this isomorphism,

$$\begin{aligned}\operatorname{tr}(\Phi(\sum_{g\in G} \lambda_g g)) &= \langle [\Phi(\sum_{g\in G} \lambda_g g)](e), e\rangle \\ &= \langle \sum_{g\in G} \lambda_g g, e\rangle \\ &= \lambda_e.\end{aligned}$$

□

We can apply the same mechanics on a more interesting example:

**Exercise 2.** *Construct a model for the Von Neumann Algebra when $G = \mathbb{Z}^n$.*

Let us do the $n = 1$ case, as the $n > 1$ cases are similar. Let $z \in \ell^2(\mathbb{Z})$, $z = \sum_{m\in\mathbb{Z}} \hat{f}(m)\cdot m$ with $\|z\|^2_{\ell^2(\mathbb{Z})} = \langle z, z\rangle = \sum_{m\in\mathbb{Z}} |\hat{f}(m)|^2 < \infty$. Now, we may identify $m \mapsto e^{imx}$. Indeed, $m$ acts as an orthonormal basis element of the Hilbert space $\ell^2(\mathbb{Z})$, whereas $e^{imx}$ acts as an orthonormal basis element of the Fourier space with inner product

$$\langle e^{im_1x}, e^{im_2x}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im_1x}e^{im_2x}dx,$$

and one can verify that $\{e^{imx}\}_{m\in\mathbb{Z}}$ is indeed an orthonormal basis. We denote this identification $\varphi$.

Let $T^1$ denote the one-dimensional torus. Then, Plancherel's theorem tells us that the function $f : T^1 \to \mathbb{C}$,

$$f(x) := \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx}$$

is isometric to its representative $\sum_{m \in \mathbb{Z}} \hat{f}(m) \cdot m$, in the sense that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{m \in \mathbb{Z}} |\hat{f}(m)|^2$$

or in other words,

$$\|f\|_2 = \|z\|_{\ell^2(\mathbb{Z})} < \infty$$

where $\| \cdot \|_2$ represents the $L^2$-norm on the torus. Therefore, the identification $\varphi$ extends linearly to a map

$$\varphi : \ell^2(\mathbb{Z}) \to L^2(T^1),$$

which is known to be bijective by Plancherel. Thus, $\varphi$ is an isometric isomorphism of Hilbert spaces $\ell^2(\mathbb{Z}) \to L^2(T^1)$.

Moreover, if $k \in \mathbb{Z}$. Then, $\varphi(kz) = e^{ikx} \cdot \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx}$, so we can define the action of $k$ on $L^2(T^1)$ to be left multiplication with $e^{ikx}$ to obtain that $\varphi$ is an $\mathbb{Z}$-equivariant isometric isomorphism of Hilbert spaces.

Under this identification,

$$\mathcal{N}(\mathbb{Z}) = \mathcal{B}(L^2(T^1))^{\mathbb{Z}}.$$

We now claim that $\mathcal{B}(L^2(T^1))^{\mathbb{Z}} \cong L^\infty(T^1)$, where the isomorphism is given by

$$M_g : L^2(T^1) \to L^2(T^1),$$

$$[M_g(f)](x) = g(x)f(x).$$

and $L^\infty(T^1)$ is the space of function on $T^1$ with finite essential supremum.

Let $g \in L^\infty(T^1)$, $k \in \mathbb{Z}$, $\lambda_1, \lambda_2$ scalars. Then

$$\begin{aligned}
&[M_g(k * f)](x) = g(x)e^{ikx} f(x) = e^{ikx} g(x) f(x) = [k * M_g(f)](x) \\
&[M_g(\lambda_1 f_1 + \lambda_2 f_2)](x) = \lambda_1 g(x) f_1(x) + \lambda_2 g(x) f(x) = [\lambda_1 M_g(f_1) + \lambda_2 M_g(f_2)](x) \\
&\|M_g(f)\|_2^2 = \|gf\|_2^2 \leq \|g\|_\infty \|f\|_2^2 \Rightarrow \|M_g\|_{\text{op}} \leq \|g\|_\infty < \infty
\end{aligned}$$

so $M_g$ is a $\mathbb{Z}$-equivariant bounded operator $L^2(T^1) \to L^2(T^1)$.

Now let $M \in \mathcal{B}(L^2(T^1))^{\mathbb{Z}}$. Then, if $f \in L^2(T)$, we may write $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}(n) e^{ikx}$. Then,

$$[M(f)](x) = [M(\sum_{k \in \mathbb{Z}} \hat{f}(k) k * 1)](x) = [\sum_{k \in \mathbb{Z}} \hat{f}(k) k * M(1)](x) = f(x)[M(1)](x),$$

almost everywhere. Then,

$$M(f) = M(1)f$$

and it remains to show that $M(1) \in L^\infty(T^1)$. Since $M(f) \in L^2(T^1)$, we have

$$\|M(f)\|_2 = \|M(1)f\|_2 \leq \|M\|_{\text{op}} \|f\|_2 < \infty$$

Now, suppose $\|M(1)\|_\infty = \infty$. Then, by definition, $M(1) = \infty$ on a set $A$ of measure non-zero. Then, letting $f = 1$, we have the following:

$$\|M\|_{\text{op}} \geq \|M(1)\|_2 = \frac{1}{2\pi}\int |M(1)(x)|^2 dx \geq \frac{1}{2\pi}\int_A \infty \cdot dx = \infty,$$

a contradiction. Therefore, $M(1) \in L^\infty(T^1)$. Under the identification $M_g \mapsto g$, the trace is for $g \in L^\infty(T^1)$,

$$\text{tr}_{\mathcal{B}(L^2(T^1))^{\mathbb{Z}}}(M_g) = \langle M_g(1), 1\rangle_{L^2(T^1)} = \langle g, 1\rangle_{L^2(T^1)} = \frac{1}{2\pi}\int g(x)dx.$$

# 3 Hilbert Module

We start with this rather heavy definition:

**Definition 7** (Hilbert Module)**.** *A Hilbert $\mathcal{N}(G)$-module $(V, G)$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists a Hilbert space $H$ and an isometric linear $G$-embedding of $V$ into the tensor product of Hilbert spaces $H \otimes \ell^2(G)$ with the inherited $G$-action acting on $\ell^2(G)$. A map of Hilbert $\mathcal{N}(G)$-modules $f : V \to W$ is a bounded $G$-equivariant operator.*

Let's wrap our heads around this finite case first:

**Definition 8** (Finitely Generated Hilbert Module)**.** *We call a Hilbert $\mathcal{N}(G)$-module $V$* finitely generated *if there is an $n \geq 0$ and a surjection $\oplus_{i=1}^n \ell^2(G) \to V$ of Hilbert $\mathcal{N}(G)$-modules.*

It turns out that an $\mathcal{N}(G)$-module is well-understood when $G$ is finite:

**Exercise 3.** *If $G$ is finite, show that a finitely generated Hilbert $\mathcal{N}(G)$-module is the same as a finite-dimensional unitary representation of $G$.*

*Proof.* If $G$ is finite, $\ell^2(G) = \mathbb{C}G$.

($\Rightarrow$) Let $g \in G$. $\varphi : \oplus_{i=1}^n \mathbb{C}G \to V$ be the surjection of $\mathcal{N}(G)$-modules. Then, every element $v \in V$ can be written as $\varphi(\sum_{g\in G}\lambda_g^1 g, \ldots, \sum_{g\in G}\lambda_g^n g)$, and since $\varphi$ is an equivariant operator, we have

$$\varphi(\sum_{g\in G}\lambda_g^1 g, \ldots, \sum_{g\in G}\lambda_g^n g) = (\sum_{g\in G} g) * \varphi(\lambda_g^1, \ldots, \lambda_g^n),$$

where the action of each $g \in G$ is linear isometric, and thus unitary. Thus, the surjective homomorphism $(\sum_{g\in G}\lambda_g^1 g, \ldots, \sum_{g\in G}\lambda_g^n g) \mapsto (\sum_{g\in G} g) * \varphi(\lambda_g^1, \ldots, \lambda_g^n)$ is a map from $G$ to its representation on $V = \text{Span}\{\varphi(\lambda_g^1, \ldots, \lambda_g^n) : g \in G\}$.

($\Leftarrow$) Let $U(G)$ be a unitary representation of $G$ of dimension $n$. Let $V$ be the $n$-dimensional Hilbert space on $\mathbb{C}$ that $U(G)$ acts on, with the standard inner product. Then

$G$ acts linearly and isometrically on $V$. The goal is to make a surjection $\varphi : \oplus_{i=1}^n \mathbb{C}G \to V$ of $\mathcal{N}(G)$-modules. Let $\{v_i\}_{i=1}^n$ be a basis for $V$. For every $(\sum_{g\in G} \lambda_g^1 g, \ldots, \sum_{g\in G} \lambda_g^n g)$, define

$$\varphi(\sum_{g\in G} \lambda_g^1 g, \ldots, \sum_{g\in G} \lambda_g^n g) := \sum_{g\in G} U(g)[\lambda_g^1 v_1 + \cdots + \lambda_g^n v_n].$$

This is a surjection because $(\lambda_e^1 e, \ldots, \lambda_e^n e) \mapsto \lambda_e^1 v_1 + \cdots + \lambda_e^n v_n$, which can be made linear by defining

$$\varphi(\sum_{g\in G} \lambda\lambda_g^1 g, \ldots, \sum_{g\in G} \lambda\lambda_g^n g) = \lambda\varphi(\sum_{g\in G} \lambda_g^1 g, \ldots, \sum_{g\in G} \lambda_g^n g).$$

Since $g$ is linear isometric on $\mathbb{C}G$, we can make the action of $\varphi$ $G$-equivariant by defining for $h \in G$,

$$\varphi(\sum_{g\in G} h\lambda_g^1 g, \ldots, \sum_{g\in G} h\lambda_g^n g) = U(h)\varphi(\sum_{g\in G} \lambda_g^1 g, \ldots, \sum_{g\in G} \lambda_g^n g).$$

$\square$

Note: the key step in this exercise was the line

$$\varphi(\sum_{g\in G} \lambda_g^1 g, \ldots, \sum_{g\in G} \lambda_g^n g) = (\sum_{g\in G} g) * \varphi(\lambda_g^1, \ldots, \lambda_g^n),$$

from which we draw the following observations:

1. The definition of finitely generate Hilbert Module matches that of the Hilbert Module when $H$ is a finite vector space of dimension $n$, because $H \otimes \ell^2(G)$ is the same as $n$ copies of $\ell^2(G)$.

2. The interaction between $H$ and $\ell^2(G)$ can be observed to be, at least in this finite case, given by the formula above: the coefficients associated to each $g \in G$ in $\ell^2(G)$ can be 'transferred' to the Hilbert space $H$ when pulling the $G$-action out of the surjection $\varphi$.

3. A $G$-module map is not necessarily an isometry, as the $G$-action inside and outside of $\varphi$ is applied separately to the domain and target.

# 4 Von Neumann Dimension

**Definition 9** (Von Neumann Dimension)**.** *The von Neumann dimension of a Hilbert $\mathcal{N}(G)$-module $V$ is defined as*

$$\dim_{\mathcal{N}(G)}(V) := tr_{\mathcal{N}(G)}(id : V \to V).$$

Surprisingly, we have also seen an example of an $\mathcal{N}(G)$-module for when $G$ is infinite:

**Exercise 4.** *Let $A \subset T^1$ be any measurable set, and let $\chi_A \in L^\infty(T^1)$ be its characteristic function. Denote by $M_{\chi_A} : L^2(T^1) \to L^2(T^1)$ the $\mathbb{Z}$-equivariant map given by multiplication with $\chi_A$. Show that its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z})$-module, with $\dim_{\mathcal{N}(\mathbb{Z})}(V) = length(A)$.*

*Proof.* As we've seen in a previous exercise about Fourier, $L^2(T^1) \cong \ell^2(\mathbb{Z})$, and since $V$ is the set of $L^2(T^1)$ functions with support in $A$, the bounded $\mathbb{Z}$-equivariant operator $M_{\chi_A}$ gives us a surjection $L^2(T^1) \to V$, where the action of $\mathbb{Z}$ is pointwise multiplication by $e^{ikx}$ for $k \in \mathbb{Z}$ for both $L^2(T^1)$ and $V$. We deduce that $V$ is a finitely generated $\mathcal{N}(G)$-module. Now, the map $\mathrm{id} : V \to V$ is simply given by $1 = \chi_A$, and therefore,

$$\mathrm{tr}_{\mathcal{N}(G)}(\mathrm{id} : V \to V) = \int_A 1 dx = \mathrm{length}(A).$$

□

# 5 Properties of the von Neumann Dimension

So far, the von Neumann dimension looks odd in various ways. Let us try to understand the dimension through its surprisingly nice properties:

**Definition 10** (Weak Exactness)**.** *We call a sequence of Hilbert $\mathcal{N}(G)$-modules*

$$0 \to U \xrightarrow{c_p} V \xrightarrow{c_{p+1}} W \to 0$$

weakly exact *if* $\ker c_{p+1} = clos(im\ c_p)$

**Theorem 1** (Properties of von Neumann Dimension)**.** *If $V$ is a Hilbert $\mathcal{N}(G)$-module, then*

1. $V = 0 \Leftrightarrow \dim_{\mathcal{N}(G)}(V) = 0.$
2. $0 \to U \to V \to W \to 0$ *is a weakly exact sequence of Hilbert $\mathcal{N}(G)$-modules, then*
$$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V)$$
3. *If $\{V_i | i \in I\}$ is a directed system of Hilbert $\mathcal{N}(G)$-submodules of $V$ directed by $\subset$, then*
$$\dim_{\mathcal{N}(G)}\left(clos\left(\bigcup_{i\in I} V_i\right)\right) = \sup_{i\in I} \dim_{\mathcal{N}(G)}(V_i)$$
4. *If $\{V_i | i \in I\}$ is a directed system of Hilbert $\mathcal{N}(G)$-submodules of $V$ directed by $\supset$, then*
$$\dim_{\mathcal{N}(G)}\left(\bigcap_{i\in I} V_i\right) = \inf_{i\in I} \dim_{\mathcal{N}(G)}(V_i)$$
5. *If $U_1$ is a Hilbert $\mathcal{N}(G)$-module, and $U_2$ be a Hilbert $\mathcal{N}(H)$-module. Then the Hilbert tensor product $U_1 \otimes U_2$ is a Hilbert $\mathcal{N}(G \times H)$-module, and*
$$\dim_{\mathcal{N}(G\times H)}(U_1 \otimes U_2) = \dim_{\mathcal{N}(G)}(U_1) \cdot \dim_{\mathcal{N}(H)}(U_2).$$

6. *Let $H \subset G$ be a subgroup of $G$ of finite index $[G : H]$. Let $V$ be the Hilbert $\mathcal{N}(G)$-module. Let, $res(V)$ be the restriction of $V$ to the Hilbert module $\mathcal{N}(H)$. If $V$ is finitely generated, then $res(V)$ is finitely generated and we have*

$$\dim_{\mathcal{N}(H)}(res(V)) = [G : H] \cdot \dim_{\mathcal{N}(G)}(V).$$

Remarks:

1. Agrees with our notion of 0-dimension.
2. Can be viewed as some form of rank-nullity theorem of weakly exact sequence, since $U$ is isomorphic to the image of $V$, and $W$ is isomorphic to the kernel of $V$ under the exact sequence.
3. and 4. Tells us that the dimension is continuous in some sense.
5. Agrees with the notion of a tensor product for vector spaces.
6. Is analogous to changing underlying fields in a vector space. Indeed, it is well-known that a vector space $V$ over $\mathbb{C}$ will assume twice its dimension over $\mathbb{R}$. $\mathcal{N}(G)$ and $\mathcal{N}(H)$ act as $\mathbb{C}$ and $\mathbb{R}$ respectively, with the multiplicative factor of the dimension being $[G : H]$.

We will be interested in computing the dimension for the following sequence of objects:

# 6 Hilbert Chain Complexes

**Definition 11** (Hilbert Chain Complex)**.** *A Hilbert $\mathcal{N}(G)$-chain complex $C_*$ is a sequence indexed by $p \in \mathbb{Z}$ of maps of Hilbert $\mathcal{N}(G)$-modules*

$$\dots \xrightarrow{c_{p+2}} C_{p+1} \xrightarrow{c_{p+1}} C_p \xrightarrow{c_p} C_{p-1} \xrightarrow{c_{p-1}} \dots$$

*such that im $c_p \subset \ker c_{p+1}$ for all $p \in \mathbb{Z}$.*

# 7 $L^2$-homology, and $L^2$-Betti numbers

**Definition 12** ($L^2$-homology and $L^2$-Betti numbers)**.** *Define the reduced p-th $L^2$-homology, and the p-th $L^2$-Betti number of a Hilbert $\mathcal{N}(G)$-chain complex $C_*$ by*

$$H_p^{(2)}(C_*) := \ker(c_p)/clos(im\ (c_{p+1}))$$

$$b_p^{(2)}(C_*) := \dim_{\mathcal{N}(G)}(H_p^{(2)}(C_*)).$$

It is well-known that if $V, W$ are Hilbert spaces, and $W$ is closed, then $V/W$ is again a Hilbert space, isomorphic to $V \cap W^{\perp}$. Hence, $H^{(2)}(C_*)$ is the Hilbert space

$$H_p^{(2)} = \ker(c_p) \cap \text{clos}(\text{im } (c_{p+1}))^{\perp}.$$

This equality is handy because we quickly see that $H_p^{(2)}(C_*)$ is an $\mathcal{N}(G)$-module: as a subspace of $C_p$, we inherit the linear isometric $G$-action from $C_p$. The embedding of $H_p^{(2)}(C_*)$ to a tensor space $H \otimes \ell^2(G)$ is simply the embedding of $C_p$ restricted to $H_p^{(2)}(C_*)$.

Thus, the $L^2$-Betti number is well-defined for all $p$.

# References

Wolfgang Lück. *$L^2$-Invariants: Theory and Applications to Geometry and K-Theory*. Springer-Verlag Berlin Heidelberg, 2002.