Overview of the classification of tripartite entanglement under SLOCC

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The goal of this document is to give the undergraduate reader an overview of tripartite entanglement under SLOCC, with no background assumed.

Warning: juvenilia!

1 Stuff that has been helpful to start reading this:

Nielsen and Chuang, especially the section of multipartite states, Bob Coecke's view of tensors, Prakash.

But really: figure out what a tensor product is in terms of vector spaces, know your linear algebra and your quantum mechanics and you're good!

2 Useful Theorems and Lemmas

Theorem 1 (Schmidt Decomposition)**.** Let $|\psi\rangle$ be a pure state of the composite system $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}$ denotes a Hilbert space. Then there exist orthonormal states $|i_A\rangle$ for system A and $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_{i=0}^{n} \lambda_i \, |i_A\rangle \, |i_B\rangle$$

where $n \leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$, and λ_i are non-negative real numbers with $\sum_{i=0}^{n} \lambda_i^2 = 1$.

Proof . First assume that system A and B are of the same dimension (we will do the general case later). Regard the coefficients a_{ij} in

$$|\psi\rangle = \sum_{i,j=0}^{m} a_{ij} |i\rangle |j\rangle \, , m = \dim \mathcal{H}_A$$

as entries in the matrix $[A]_{ij} = a_{ij}$. Then, by singular value decomposition,

$$A = UDV$$

where U and V are unitary matrices of dimension $(m+1) \times (m+1)$ and D is a diagonal matrix of dimension $(m+1) \times (m+1)$ also. Since $[AB]_{ij} = \sum_{r=0}^{m} a_{ir} b_{rj}$, we have

$$a_{ij} = \sum_{r,k=0}^{m} u_{ir} d_{rk} v_{kj}$$

plugging in our result,

$$|\psi\rangle = \sum_{i,j=0}^{m} \sum_{r,k=0}^{m} u_{ir} d_{rk} v_{kj} |i\rangle |j\rangle$$

and noting that non-diagonal entries in D are zero,

$$= \sum_{i,j=0}^{m} \sum_{k=0}^{m} u_{ik} d_{kk} v_{kj} |i\rangle |j\rangle$$

we obtain

$$= \sum_{k=0}^{m} d_{kk} \left(\sum_{i=0}^{m} u_{ik} |i\rangle \right) \left(\sum_{j=0}^{m} v_{kj} |j\rangle \right)$$

Since unitary matrices send orthonormal basis vectors to orthonormal basis vectors (here we are basically 'passing' our basis vectors $|i\rangle$ through a column of the unitary matrix), we can label the new orthonormal basis vectors as $|k\rangle$

$$= \sum_{k=0}^{m} d_{kk} |k_A\rangle |k_B\rangle$$

By the singular value decomposition, we have that the diagonal entires must be non-zero, which completes our proof for the case of $\mathcal{H}_A$ and $\mathcal{H}_B$ having the same dimension.

Without loss of generality, consider $\mathcal{H}_A$ having a greater dimension than $\mathcal{H}_B$, and let $\dim(\mathcal{H}_A) = m$, $\dim(\mathcal{H}_B) = n$. Then, make an $m \times m$ matrix using the coefficients a_{ij} of $|\psi\rangle$ 'padding' with zero entries in the last $m-n$ rows of the matrix. Since unitary transformations preserve matrix rank, the matrix obtained under singular value decomposition will be of rank n. Thus, after change of basis of the two quits, we get at most n non-zero coefficients d_{kk}, where $n \leq \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$. ∎

Corollary 2. A state of a composite system AB is a product state $\Leftrightarrow$ it has Schmidt number 1. Where the Schmidt number is the number of non-zero λ_i. Thus, $|\psi\rangle$ is a product state $\Leftrightarrow$ ρ_A and ρ_B is of rank 1.

Proof (. $\Leftarrow$)If $|\psi\rangle$ has Schmidt number 1, then it is clearly a product state. Let $|\psi\rangle = |i\rangle\,|i\rangle$. Then,

$$\rho = |\psi\rangle\,\langle\psi| = |i\rangle\,|i\rangle\,\langle i|\,\langle i|$$

By partial tracing over A,

$$\rho_A = |i\rangle\,\langle i|$$

and likewise for B. We conclude $r(\rho_A) = r(\rho_B) = 1$

($\Rightarrow$) If $\rho_A = |i\rangle\,\langle i|\,, \rho_B = |i\rangle\,\langle i|$ then, $|\psi\rangle = |i\rangle\,|i\rangle$. ∎

Meh. Basically, the partial trace of mixed states must be of Schmidt number greater than 1.

3 Three-qubits can be maximally entangled in two inequivalent ways

Theorem 3. $|\psi\rangle$ and $|\phi\rangle$ belong to the same class under SLOCC $\Leftrightarrow$ they are related by means of an invertible local operator

Lemma 4. Let us denote the tensor product by $\otimes$. If the bipartite vectors $|\psi\rangle$ and $|\phi\rangle \in \mathcal{H}^n \otimes \mathcal{H}^m$ satisfy:

$$|\phi\rangle = T_A \otimes 1_B\,|\psi\rangle$$

where T_A is a linear operator on the system A. Then the ranks r of the corresponding reduced density matrices ρ satisfy:

$$r(\rho_A^{\psi}) \geq r(\rho_A^{\phi}) \qquad r(\rho_B^{\psi}) \geq r(\rho_B^{\phi})$$

Proof . Consider the Schmidt decomposition of $|\psi\rangle$

$$|\psi\rangle = \sum_{i=0}^{n_\psi} \sqrt{\lambda_i^\psi}\, |i\rangle\, |i\rangle \qquad \sum_{i=0}^{n^\psi} \lambda_i^\psi = 1, \quad \lambda_i^\psi > 0, \quad n^\psi \leq \min(n, m)$$

with n and m being the respective dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$. We can write the operator T_A as

$$T_A = \sum_{i=0}^{n} |\mu_i\rangle \langle i|$$

where the $|\mu_i\rangle$ are vectors in $\mathcal{H}_A$. Then,

$$\rho_A^\psi = \mathrm{tr}_B \left[\left(\sum_{i=0}^{n^\psi} \sqrt{\lambda_i}\, |i\rangle\, |i\rangle \right) \left(\sum_{i=0}^{n^\psi} \langle j|\, \langle j| \right) \right]$$

$$= \sum_{k=0}^{n^\psi} \langle k| \left[\left(\sum_{i=0}^{n^\psi} \sqrt{\lambda_i}\, |i\rangle\, |i\rangle \right) \left(\sum_{i=0}^{n^\psi} \lambda_j\, \langle j|\, \langle j| \right) \right] |k\rangle$$

$$= \sum_{k=0}^{n^\psi} \lambda_k\, |k\rangle \langle k| \qquad \lambda_k > 0$$

Which is pretty awesome! Now, let $\dagger$ denote adjoint.

$$\rho_A^\phi = T_A \rho_A^\psi T_A^\dagger$$

$$= T_A \sum_{i=0}^{n^\psi} \lambda_i\, |i\rangle \langle i|\, T_A^\dagger$$

$$= \sum_{i=0}^{n^\psi} \lambda_i\, |\mu_i\rangle \langle \mu_i|$$

Which implies

$$r(\rho_A^\phi) \leq n^\psi = r(\rho_A^\psi)$$

with equality on the left $\Leftrightarrow$ T_A is an invertible operator. Because taking the partial trace of the state is completely symmetric in A and B (this is easily seen by taking the Schmidt decomposition of the state, as we did), we also have

$$r(\rho_B^\phi) \leq n^\psi = r(\rho_B^\psi)$$

as claimed. ∎

Corollary 5. If the vectors $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \cdots \otimes \mathcal{H}_N$ are connected by a local operator as

$$|\phi\rangle = T_A \otimes T_B \otimes \cdots \otimes T_N \, |\psi\rangle$$

then the locals ranks satisfy

$$r(\rho_\kappa^\psi) \geq r(\rho_\kappa^\phi), \kappa = A, B, \ldots, N$$

For each party, we can view the operator $T_A \otimes T_B \otimes \cdots \otimes T_N$ as the composition of two local operators, $T_A \otimes 1_B ... N$ and $1_A \otimes (T_B \otimes ... \otimes N)$. Using our previous lemma, $r(\rho_A^\psi) = r(\rho_A^\phi)$ and $r(\rho_{B...N}^\psi) = r(\rho_{B...N}^\phi)$. We note our choice in the splitting $T_A \otimes T_B \otimes \cdots \otimes T_N$ into two operators is completely arbitrary, meaning the previous lemma holds for any splitting of systems. We conclude that all partial ranks must then be equal[1]. In short, if $|\psi\rangle$ and $|\phi\rangle$ belong to the same class under SLOCC $\Leftrightarrow$ they are related by means of an invertible local operator.

($\Rightarrow$) The equivalence of $|\psi\rangle$ and $|\phi\rangle$ under SLOCC implies that the operator relating the two states can always be chosen to be invertible

We can think of any local operation as a serious of rounds of operations. In each round, a given party manipulates its subsystem using a local operator and communicates the result of its operation (if it included a measurement) to the rest of the party.

For example, say Alice, Bob and Carl are together in a room where a three-qubit state $|\psi\rangle$ is created. They all know what the state is, for whatever reason. Then, they each take a qubit from the state, and move arbitrarily far away from each other. Alice, looking at her state, sees only $\rho_A = \mathrm{tr}_{BC}\,\rho$, where $\rho = |\psi\rangle \langle\psi|$.

Let's say Alice now applies a POVM defined by $\sqrt{p_A} T_A$ and $\sqrt{1_A - p_A T_A^\dagger T_A}$ where p_A is the probability of the first outcome being successful. Suppose Alice obtains the state:

$$\frac{T_A \rho_A T_A^\dagger}{\mathrm{tr}(T_A^\dagger T_A \rho_A)}$$

[1]We also conclude we probably need a more rigorous proof. Because although convincing, I don't completely buy the argument yet. Whoops, I accidentally broke the fourth wall!

and communicates her result to Bob and Carl via classical channel. Bob and Carl do nothing to their qubit. Then, because Alice has communicated her result, all three parties know that the new state is:

$$T_A \otimes 1_B \otimes 1_C |\psi\rangle$$

Say Bob and Carl do exactly what Alice did, and that the resulting tripartite state is:

$$T_A \otimes T_B \otimes T_C |\psi\rangle .$$

Notice how, despite the protocol having split into several branches (ours being the one with probability of success $p_A p_B p_C$), we only needed to focus on one of those branches. In other words, if the state $|\psi\rangle$ can be locally converted into the state $|\phi\rangle$ with a non-zero probability, then at least one branch does the converting job. Since we are concerned only with pure states, we can always characterize mathematically the relevant branch as an operator which factors out as the tensor product of a local operator for each party.

We are ready to tackle the the $\Rightarrow$ direction of the proof:

Proof $\Rightarrow$. Since the composition of invertible operators are invertible, we may without loss of generality consider the pair $|\phi\rangle$ and $|\psi\rangle$:

$$|\phi\rangle = T_A \otimes 1_{B\dots N} |\psi\rangle .$$

We write both states in their Schmidt decomposition,

$$|\psi\rangle = \sum_{i=0}^{n^\psi} \sqrt{\lambda_i^\psi} |i\rangle |\tau_i\rangle$$

$$|\phi\rangle = \sum_{i=0}^{n^\phi} \sqrt{\lambda_i^\phi} |i'\rangle |\tau_i\rangle .$$

Let U_A be the unitary relating $|i\rangle$ to $|i'\rangle$ in the following way:

$$|\phi\rangle = \sum_{i=0}^{n^\phi} \sqrt{\lambda_i^\phi} (U_A |i\rangle) |\tau_i\rangle .$$

By our lemma, $r(\rho_A^\psi) \geq r(\rho_A^\phi)$ and $r(\rho_B^\psi) = r(\rho_B^\phi) \Rightarrow r(\rho_A^\psi) = r(\rho_A^\phi)$ or equivalently $n^\psi = n^\phi$.

A quick check will confirm that T_A can be written as

$$T_A = U_A(T_{A1} + T_{A2})$$

where

$$T_{A1} = \sum_{i=0}^{n^\psi} \sqrt{\lambda_i^\phi / \lambda_i^\psi} \left|i\right\rangle \left\langle i\right|$$

$$T_{A2} = \sum_{n^\psi+1}^{n} \left|\mu_i\right\rangle \left\langle i\right|$$

and the $|\mu_i\rangle$ are vectors in $\mathcal{H}_A$.

Since $T_{A2} \otimes 1_{B...N} |\psi\rangle = 0$, we are free to define an T'_{A2} such that $T_{A1} + T'_{A2}$, say

$$T'_{A2} \equiv \sum_{n^\psi+1}^{n} \left|i\right\rangle \left\langle i\right|$$

Now, let

$$T'_A \equiv T_{A1} + T'_{A2}$$

then we have

$$|\phi\rangle = T'_A \otimes 1_{B...N} |\psi\rangle$$

with $T'_A \otimes 1_{B...N}$ invertible as wanted. ∎

Proof $\Leftarrow$. Recall the statement:
$|\phi\rangle = T_A \otimes T_B \otimes \cdots \otimes T_N |\psi\rangle$ implies $|\phi\rangle$ and $|\psi\rangle$ are SLOCC equivalent.

If $|\phi\rangle = T_A \otimes T_B \otimes \cdots \otimes T_N |\psi\rangle$, then a local protocol exists for the parties to transform $|\psi\rangle$ into $|\phi\rangle$ with a non-zero probability of success. Recall that party A (Alice) has to apply a POVM defined by operators $\{\sqrt{p_A}A, \sqrt{1_A - p_A A^\dagger A}\}$, where p_A is the probability that Alice successfully performs her protocol, such that $p_A A^\dagger A \leq 1_A$. We define the other parties' POVM similarly. Then, the probability of success of converting $|\psi\rangle$ into $|\phi\rangle$ is $p_A p_B \cdots p_N > 0$. ∎

3.1 Classifying three-qubit entanglement: the trivial cases

We are first going to classify three-qubits in terms of their partial ranks. We are working with the states in the space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, with $\dim(\mathcal{H}_A) = n_A, \dim(\mathcal{H}_B) = n_B, \dim(\mathcal{H}_C) = n_C$ respectively.
Proposition 6. The ranks of the partial traces are invariant under SLOCC.

Proof . Consider the $\rho_A = \mathrm{tr}_{BC}(\rho)$, where ρ is a pure state three-qubit density matrix. Since ρ_A is a positive operator, it has spectral decomposition:

$$\rho_A = \sum_{i=0}^{n_A-1} \alpha_i |i\rangle \langle i| \qquad \alpha_i \in \mathbb{R}$$

A SLOCC operation T_A on ρ_A is invertible. Thus, a SLOCC operation makes the following mapping:

$$\rho_A \mapsto K \sum_{i=0}^{n_A-1} \alpha_i T_A |i\rangle \langle i| T_A^\dagger$$

where K is a normalizing constant. Since T_A is an invertible operator, it sends a set of basis vectors to another set of basis vectors injectively. Thus the set $\{T_A |i\rangle\}$ is also a basis. If we look at T_A acting on kets $|i\rangle$ and $T_A^\dagger$ acting on bras $\langle i|$, we can easily observe that the rank of ρ_A is preserved under SLOCC. ∎

3.1.1 Class A-B-C (product states)

$$r(\rho_A) = r(\rho_B) = r(\rho_C) = 1.$$

We take representative

$$|\psi\rangle = |0\rangle |0\rangle |0\rangle$$

We can obtain any other three-qubit product state by acting on $|\psi\rangle$ with some local unitary (LU), $U_A \otimes U_B \otimes U_C$.

3.1.2 Classes A-BC, AB-C, C-AB

A-BC stands for the B and C qubits being entangled. Recalling that the partial trace of an entangled qubit is a mixed state, and that $\dim(\mathcal{H}_B) = \dim(\mathcal{H}_C) = 2$,

$$r(\rho_A) = 1, r(\rho_B) = r(\rho_C) = 2$$

Using the Schmidt decomposition for two-qubits, we may obtain up to LU

$$|\psi_{A-BC}\rangle = |0\rangle (\cos\delta |00\rangle + \sin\delta |11\rangle) \qquad \cos\delta \geq \sin\delta > 0, \quad \delta \in [0, \pi/2]$$

We can then obtain up to local invertible operator (ILO),

$$|\psi_{A-BC}\rangle = \frac{1}{\sqrt{2}} |0\rangle (|00\rangle + |11\rangle)$$

3.2 Classifying three-qubit entanglement: the tripartite entanglement cases

Proposition 7. ILOs cannot change the minimum number of terms in a state.

Proof . Take for example the state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and consider the ILO $T_A \otimes T_B \otimes T_C$. Then,

$$T_A \otimes T_B \otimes T_C\,|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|T_A 0\rangle\,|T_B 0\rangle\,|T_C 0\rangle + |T_A 1\rangle\,|T_B 1\rangle\,|T_C 1\rangle\right)$$

By invertibility of our operator, $|T_A 0\rangle$ and $|T_A 1\rangle$ must be linearly independent. Therefore, there is no way to reduce the minimal number of terms needed to describe a state under SLOCC. ■

Lemma 8. Let $\sum_{i=1}^{l} |e_i\rangle\,|f_i\rangle$ be a product decomposition for the state $|\eta \in \mathcal{H}_E \mathcal{H}_F\rangle$. Then the set of states $\{|e_i\rangle\}_{i=1}^{l}$ span the range of $\rho_E = \text{tr}_F\,|\eta\rangle\,\langle\eta|$.

Proof . Using the definition of the partial trace, $\rho_E = \sum_{i=1}^{l} \langle f_i \,|\, f_j\rangle\,|e_j\rangle\,\langle e_j|$. $|\nu\rangle$ is in the range of $\rho_E \iff$ there exists $|\mu\rangle$ such that $|\nu\rangle = \rho_E\,|\mu\rangle$. Well, assuming that such $|\mu\rangle$ exists, we have $|\nu\rangle = \sum_{i,j=1}^{l} \langle f_i \,|\, f_j\rangle\,\langle e_i \,|\, \mu\rangle\,|e_j\rangle$, where the two first coefficients are simply scalars in $\mathbb{C}$. ■

Remark 9. To continue, you should read Theorem 1 and 2 of the paper "Local description of quantum inseparability"[] because the classification is pretty much based on those two theorems.

Alright, if you feel like cheating here are the statements of the two theorems for your convenience. *Ahem* here's a recall of the theorems to facilitate reading, is what I meant...

Theorem 10. *Given a plane $P_1 \subset \mathbb{C}^2 \otimes \mathbb{C}^2$, generated by two product vectors $v_1 = v_1^1 \otimes v_1^2$ and $v_2 = v_2^1 \otimes v_2^2$, either every vector in the plane is a product vector or only the generating vectors v_1 and v_2 are product vectors, up to scalar multiples.*

Theorem 11. *Any plane P_2 in $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product vector.*

Copy pasted for your convenience. But really, go check out the proofs! They appear in Section 4.

Proposition 12. $R(\rho_{BC})$ is of dimension 2, where R denotes the range.

Proof . Using the Schmidt decomposition on $|\psi\rangle \in \mathbb{C}^2 \otimes \left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$ we get:

$$|\psi\rangle = \sum_{i=0}^{n-1} \lambda_i |i_A\rangle |i_{BC}\rangle \qquad n \leq 2$$

But since we know that $R(\rho_A)$ is of dimension 2, with $\rho = |\psi\rangle\langle\psi|$, n cannot be 1, so it must be 2. Hence,

$$\rho = \lambda_0^2 |00\rangle\langle 00| + \lambda_0 \bar{\lambda_1} |00\rangle\langle 11| + \bar{\lambda_0}\lambda_1 |11\rangle\langle 00| + \lambda_1^2 |11\rangle\langle 11|$$

Partial tracing in A ,

$$\rho_{BC} = \lambda_0^2 |0\rangle\langle 0| + \lambda_1^2 |1\rangle\langle 1|$$

which implies ρ_{BC} is of dimension 2.

Now that we know ρ_{BC} is a plane in $\mathbb{C}^2 \otimes \mathbb{C}^2$, we can use Theorem 10 and Theorem 11 to conclude that the range of our entangled states contain either one or two product vectors, up to scalar multiples. If we had infinitely many product vectors in the range, we could pick a basis such that the state itself is a product vector. But we know that the state is entangled, and hence it is impossible for the state to be a product vector. ∎

Remark 13. I didn't get this at first; here was a proof by example that the person latexing isn't all-knowing by any means.

Proposition 14. ILOs take product vectors to product vectors

Proof . Consider $|u_1\rangle |u_2\rangle$ acted on by the ILO $T_A \otimes T_B$. Then, we get $|T_A u_1\rangle |T_B u_2\rangle \neq 0$, since the operator is invertible. It is also impossible to "add" more minimal terms to this product decomposition. So, since we can't add nor subtract terms to product vectors using ILOs, ILOs must take product vectors to product vectors. ∎

Corollary 15. The number of product vectors in $R(\rho_B C)$ is invariant under ILO.

This means we can partition the equivalence class of three-qubit entangled states into 1) states that have two product vectors in their range, 2) states that have one product vector in their range. We're almost there!

3.2.1 GHZ class

Suppose $R(\rho_{BC})$ contains, up to scalar multiples, two product vectors, $|b_1\rangle |c_1\rangle$ and $|b_2\rangle |c_2\rangle$. We claim we can "uniquely" decompose $|\psi\rangle$ as

$$|\psi\rangle = |a_1\rangle |b_1\rangle |c_1\rangle + |a_2\rangle |b_2\rangle |c_2\rangle$$

with

$$|a_1\rangle = \langle \xi_1 \,|\, \psi \rangle\,, \quad |a_2\rangle = \langle \xi_2 \,|\, \psi \rangle\,, \qquad \langle \xi_1 \,|\, b_2 c_2 \rangle = \langle \xi_2 \,|\, b_1 c_1 \rangle = 0$$

Remark 16. Oh gosh, I wish I had the LaTeXskills to draw a diagram! In the meantime you'll have to contend with my algebraic explanation and my fun putting in the LaTeXsymbol for eye amusement. And my lack of knowledge of how to put a space after LaTeX, apparently.

Proof . Because we are working in a finite dimensional state-space (eight-dimensional to be exact), we may write:

$$|\psi\rangle = \sum_{i=1}^{n} |\alpha_i\rangle \, |b_i c_i\rangle\,, \qquad n \leq 8$$

But recalling that the $|b_i c_i\rangle$ are in a 2-dimension subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$|b_i c_i\rangle = \gamma_i \, |b_1 c_1\rangle + \delta_i \, |b_2 c_2\rangle$$

Now,

$$\langle \xi_1 \,|\, \psi \rangle = \sum_{i=1}^{n} k \gamma_i \, |\alpha_i\rangle \equiv |a_1\rangle\,, \qquad k = \langle \xi_1 \,|\, b_1 c_1 \rangle$$

And similarly for $|a_2\rangle$. [2]

Seeing that we have a two-term product decomposition for $|\psi\rangle$, we can get:

$$|\psi_{GHZ}\rangle = \sqrt{K} \left(\cos\delta \, |000\rangle + \sin\delta e^{i\phi} \, |\varphi_A\rangle \, |\varphi_B\rangle \, |\varphi_C\rangle \right)$$

where we have taken $|a_1 b_1 c_1\rangle$ to $|000\rangle$, $\cos\delta$ and $\sin\delta$ are for normalization. Up to overall phase, $\delta \in (0, \pi/4]$, $\phi \in (0, \pi/2]$ and

$$|\varphi_A\rangle = \cos\alpha \, |0\rangle + \sin\alpha \, |1\rangle$$
$$|\varphi_B\rangle = \cos\beta \, |0\rangle + \sin\beta \, |1\rangle$$
$$|\varphi_C\rangle = \cos\gamma \, |0\rangle + \sin\gamma \, |1\rangle$$

where $\alpha, \beta, \gamma \in [0, 2\pi]$, for rotation reasons I'm not totally certain of. Will think about it later and update this (hopefully).

And $K = (1 + 2\cos\delta \sin\delta \cos\alpha \cos\beta \cos\phi)^{-1}$. [3] ∎

[2] Note the decomposition is, in fact, not unique. We can use $\langle -\xi_i \,|\, \psi \rangle$ to get $|-a_i\rangle$, but that doesn't change the result of the paper so we'll let it go. (Because they are physicists.)

[3] I didn't check this one.

We call this class the GHZ class because it can be checked that applying the ILO

$$\sqrt{2K}\begin{pmatrix}\cos\delta & \sin\delta\cos\alpha e^{i\phi}\\ 0 & \sin\delta\sin\alpha e^{i\phi}\end{pmatrix}\otimes\begin{pmatrix}1 & \cos\beta\\ 0 & \sin\beta\end{pmatrix}\otimes\begin{pmatrix}1 & \cos\gamma\\ 0 & \sin\gamma\end{pmatrix}$$

on

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

3.2.2 W class

Proposition 17. Suppose $R(\rho_{BC})$ contains one product vector. Then, we can "uniquely" decompose $|\psi\rangle$ into

$$|\psi_W\rangle = |a_1\rangle|b_1\rangle|c_1\rangle + |a_2\rangle|\phi_{BC}\rangle.$$

Proof . Since $|b_1c_1\rangle$ is the only product vector in the plane $\mathrm{R}(\rho_B C)$, we can always chose a vector $|\phi_B C\rangle$ orthogonal to $|b_1c_1\rangle$ that is not a product vector. Then, define:

$$|a_1\rangle = \langle\phi_{BC}\,|\,\psi\rangle, \quad |a_2\rangle = \langle b_1c_1\,|\,\psi\rangle$$

Now, taking $|b_1c_1\rangle$ into $|00\rangle$ and $|a_2\rangle = |0\rangle$ we must have

$$|\phi_{BC}\rangle = x\,|01\rangle + y\,|10\rangle + z\,|11\rangle$$

by orthogonality of $|b_1c_1\rangle$ and $|\phi_{BC}\rangle$.

We claim that $z = 0$. To see this, let

$$x\,|01\rangle + y\,|10\rangle + z\,|11\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle)(\gamma\,|0\rangle + \delta\,|1\rangle)$$

$$\Rightarrow \alpha\gamma = 0, \quad \alpha\delta = x, \quad \beta\gamma = y, \quad \beta\delta = z$$

If $\alpha = 0$, then $x = 0$ and $\beta\gamma = y, \beta\delta = z$, which is a possible product decomposition since γ and δ are degrees of freedom. Similarly for $\gamma = 0$. The only way to get an impossible product decomposition is to have $z = 0$, because that would imply that x or y is 0. This is a contradiction, since $|\phi_{BC}\rangle$ is an entangled state.

By absorbing negative signs into the kets,[4] we can obtain

$$|\psi_W\rangle = \left(\sqrt{c}\,|1\rangle + \sqrt{d}\,|0\rangle\right)|00\rangle + |0\rangle\left(\sqrt{a}\,|01\rangle + \sqrt{b}\,|10\rangle\right)$$

[4] I don't know how exactly.

or the expanded form:

$$|\psi_W\rangle = \sqrt{a}\,|001\rangle + \sqrt{b}\,|010\rangle + \sqrt{c}\,|100\rangle + \sqrt{d}\,|000\rangle$$

Since we are looking for the minimal number of product terms, and that states decomposable into two product terms belong to the GHZ class, it is very useful to note that the state

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

is (as expected) a good representative of the class since it has only 3 product terms (and hence is minimal in its number of terms). It can be checked that $|\psi_W\rangle$ can be obtained by applying the ILO

$$\begin{pmatrix} \sqrt{a} & \sqrt{d} \\ 0 & \sqrt{c} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \frac{\sqrt{3b}}{\sqrt{a}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

∎

YAY! We are pretty much done with classification!

4 Local description of quantum inseparability

Theorem 18. Given a plane $P_1 \subset \mathbb{C}^2 \otimes \mathbb{C}^2$, generated by two product vectors $v_1 = v_1^1 \otimes v_1^2$ and $v_2 = v_2^1 \otimes v_2^2$, either every vector in the plane is a product vector or only the generating vectors v_1 and v_2 are product vectors, up to scalar multiples.

Proof . By picking the correct basis, we let

$$v_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_1^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2^1 = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix}, v_2^2 = \begin{pmatrix} \cos B \\ sinB \end{pmatrix}$$

. Then, $p_1 \in P_1$ can be written

$$p_1 = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \otimes \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}$$

p_1 is separable $\iff$ $\sin A \sin B = 0$.

($\Leftarrow$) Indeed, if $\sin A = 0$, then

$$\begin{aligned} p_1 &= \alpha_1 \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \beta_1 \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}\cos B\\ \sin B\end{pmatrix} \\ &= \begin{pmatrix}1\\0\end{pmatrix} \otimes \left[\alpha_1 \begin{pmatrix}1\\0\end{pmatrix} + \beta_1 \cos A \begin{pmatrix}\cos B\\ \sin B\end{pmatrix}\right] \end{aligned}$$

and similarly for $\sin B = 0$, except we factor out the second $\begin{pmatrix}1\\0\end{pmatrix}$.

($\Rightarrow$) Suppose $\sin A \sin B \neq$ and p_1 is non-trivially separable ($\alpha_1, \beta_1 \neq 0$) Then,

$$p_1 = \alpha_1 \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \beta_1 \begin{pmatrix}\cos A\\ \sin A\end{pmatrix} \otimes \begin{pmatrix}\cos B\\ \sin B\end{pmatrix} = u_1 \otimes u_2$$

Then, under our choice of basis,

$$u_1 \otimes u_2 = \left[cos\theta \begin{pmatrix}1\\0\end{pmatrix} + sin\theta \begin{pmatrix}\cos A\\ \sin A\end{pmatrix}\right] \otimes \left[\cos\gamma \begin{pmatrix}1\\0\end{pmatrix} + \sin\gamma \begin{pmatrix}\cos B\\ \sin A\end{pmatrix}\right]$$

which implies

$$\cos\theta \cos\gamma = \alpha_1, \quad \cos\theta \sin\gamma = 0, \quad \sin\theta \sin\gamma = \beta_1, \quad \sin\theta \cos\gamma = 0$$

Contradiction. ∎

We could have also (more painfully) expanded the vectors generating P_1 into four-dimensional vectors, and look at the condition for them being separable. I did, and it gave the same conclusion (so you don't have to do it).
Theorem 19. Any plane P_2 in $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product vector, up to scalar multiples.

Proof . Consider $p_2 \in P_2$. By picking the right basis, we can write:

$$p_2 = \alpha_2 \begin{pmatrix}A\\0\\0\\B\end{pmatrix} + \beta_2 \begin{pmatrix}CB\\ \gamma\\ \delta\\ -CA\end{pmatrix}$$

where the vector is perpendicular to the first.

First, assume that none of the generating vectors are product vectors, that is:

$$AB \neq 0, \quad C^2 AB + \gamma\delta \neq 0$$

To see this, notice the first expression means that the first generating vector is in some kind of (generalized) Bell state. As for the second generating vector:

if $\begin{pmatrix} CB \\ \gamma \\ \delta \\ -CA \end{pmatrix}$ is a product, then

$$\begin{pmatrix} CB \\ \gamma \\ \delta \\ -CA \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Using $(ac)(bd) = (ad)(bc)$, we get

$$(CB)(-CA) = (\gamma)(\delta), \qquad \Rightarrow C^2AB + \gamma\delta = 0$$

Then, using the exact same logic as above on p_2 we get

$$p_2 = \begin{pmatrix} \alpha_2 A + \beta_2 CB \\ \beta_2\gamma \\ \beta_2\delta \\ \alpha_2 B - \beta_2 CA \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} \otimes \begin{pmatrix} c' \\ d' \end{pmatrix}$$

$$\Rightarrow \alpha_2^2 AB + \alpha_2\beta_2(CB - CA) - \beta_2^2(C^2AB + \gamma\delta) = 0$$

We now have a quadratic in, say, $\alpha_2 \in \mathbb{C}$, i.e., we are guaranteed to have one or two distinct solutions, hence one or two product vectors. ∎

Question I had: "But wait! Don't we have instead infinitely many solutions? Since in our quadratic we only fix α_2, β_2 is a degree of freedom, implying there are not two solutions but infinitely many!"

Answer I had (after panicking a bit): "Well, yeah, but we are guaranteed at least two solutions right? So, take those two solutions/product vectors and use them as generating vectors in theorem one." We have that either the solutions are the only solutions (i.e., $\begin{pmatrix} CB \\ \gamma \\ \delta \\ -CA \end{pmatrix}$ is just a scalar multiple of $\begin{pmatrix} A \\ 0 \\ 0 \\ B \end{pmatrix}$ and changing β_2 only "expands or contracts" our product vectors), or we have an infinity of solutions. "All is well."

Another question I had: "Yeah, okay, but what if we have only one distinct root for α_2 (i.e. what if we have only one generating product vector)? Then can't we get infinitely many valid β_2."

Answer (after panicking even more): "Yup. **Fucking physicists**[5]. Getting infinitely many valid β_2 means that we must get at least two different product vectors. This time we can't get $\begin{pmatrix} CB \\ \gamma \\ \delta \\ -CA \end{pmatrix}$ being a scalar multiple of the generating product vector because we assumed that $\begin{pmatrix} CB \\ \gamma \\ \delta \\ -CA \end{pmatrix}$ wasn't a product vector in our proof. So having different β_2's and a fixed α_2 definitively gets us at least two (okay, infinitely many) non-parallel vectors, so we have infinitely many product vectors in that case. All is well. Geez, don't you wish they told us that? I know I do!"

So in conclusion : If we have two generating product vectors, then we either have them as only product vectors in P_2 or every vector in P_2 are product vectors. If we have two generating non-product vectors, then at least one vector in P_2 that's in the plane is a product vector. The case of one product vector and one non-product vector generating the plane is exactly the same as the previous case from Theorem 2 (honestly, that one confused me for a bit), so we're done!

5 On multi-particle entanglement

We will look at the dimension of local unitary (LU) orbits using cool Lie algebra. Perhaps we can apply the tool to SLOCC as well . . . hmm . . .

5.1 Intro

Group theoretically, the space of states of n 1/2-spin particles is the n-fold tensor product $\mathbb{C}^{2n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. The group of local transformation is

[5] No wonder they have so many children.

the n-fold product $U(2)^n = U(2) \otimes \cdots \otimes U(2)$ where $U(2)$ represents unitary 2×2 matrices with complex entries. Thus, the space of orbit is:

$$\mathbb{C}^{2n}/U(2)^{2n}$$

Definition 20. We call the *dimension* of the orbit the number of parameters needed to describe the position of $|\psi\rangle$ on its orbit.

The total number of parameters describing the space of states minus the number of parameters describing a generic orbit gives the number of parameters describing the location of the orbit.

5.2 The number of parameters needed to describe inequivalent states (dimension of a general orbit)

Proposition 21. $U(2)$ is described by 4 real parameters.

Proof . Let's count!

$$U(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

where $\bar{a}$ is the complex conjugate of a. Recalling that a number times its conjugate is always real, we obtain have the following *real* equations:

$$\bar{a}a + \bar{c}c = 1$$

$$\bar{b}b + \bar{d}d = 1$$

and the following *complex* equation:

$$b\bar{a} + d\bar{c} = 0$$

the following equation being the same as the one above by conjugation on both sides:

$$\bar{b}a + \bar{d}c = 0$$

We started with 4 *complex* degrees of freedom (DOF), which is equal to 8 *real* DOF. We subtract 1 *complex* DOF and 2 *real* DOF to obtain 4 *real* DOF. We conclude the dimension of the smooth manifold represented by $U(2)$ must be 4. ∎

5.2.1 Calculating the dimension of a general orbit

Remark 22. Warning: This section is not explained properly; not because of my incompetence at explaining stuff, but because I have no idea what I'm doing.

Because $U(2)$ has dimension 4, we can use 4 Hermitian matrix basis vectors H to generate $U(2)$ using the e^i functor, since:

$$e^{(iH)\dagger} = e^{-iH} \Rightarrow e^{iH} \text{ is unitary}$$

Now let's do the calculation for a single qubit and generalize afterwards. We pick as basis (one can easily verify the following are Hermitian):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now take an element $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$ We want to look at the infinitesimal action of $e^{i\sigma_x}$. We do so by looking at the first-order Taylor expansion of $e^{i(\epsilon\sigma_x)}$:

$$e^{i(\epsilon\sigma_x)} = 1 + i\epsilon\sigma_x + \dots$$

Thus infinitesimally, we have the mapping

$$|\psi\rangle \mapsto (1 + i\epsilon\sigma_x)\,|\psi\rangle = \begin{pmatrix} \alpha + i\epsilon\beta \\ \beta + i\epsilon\alpha \end{pmatrix}$$

Now, let's represent α and β in terms of *real* parameters. Let $\alpha = c_1 + id_1, \beta = c_2 + id_2$. Then, one can check that

$$|\psi\rangle = \begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix} \mapsto \left(1 + \epsilon \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix}$$

that is, there is an induced action on a function:

$$f(c_1, d_1, c_2, d_2) \mapsto f(c_1 - \epsilon d_2, d_1 + \epsilon c_2, c_2 - \epsilon d_1, d_2 + \epsilon c_1)$$

Differentiating with respect to ϵ, we use the chain rule to obtain:

$$\left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \epsilon} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \epsilon} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial \epsilon} + \frac{\partial f}{\partial x_4}\frac{\partial x_4}{\partial \epsilon}$$

where x_i is the ith argument of the function. since

$$\frac{\partial c_1 - \epsilon d_2}{\partial c_1} = 1$$

we have

$$\partial x_1 = \partial c_1$$

and thus

$$\begin{aligned}\left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} &= \left(-d_2\frac{\partial}{\partial c_1} + c_2\frac{\partial}{\partial d_1} - d_1\frac{\partial}{\partial c_2} + c_1\frac{\partial}{\partial d_2}\right) f \\ &= u_x \nabla f\end{aligned}$$

where u_x is the directional derivative

$$u_x = \begin{pmatrix} -d_2 \\ c_2 \\ d_1 \\ c_1 \end{pmatrix}$$

and similarly for u_y, u_z, u_1. Checking for linear dependence, we find 3 out of the 4 vectors are linearly independent. Thus the dimension of the orbit is 3.

6 Three-qubit Schmidt Decomposition

We would like to Schmidt-decompose three qubits, but unfortunately using singular value decomposition on a 3×3 square matrix won't work. For one, we will typically have 8 entries for a matrix that takes 9. Also, we use 2 unitary matrices in the singular value decomposition, but we need to send 3 sets of orthonormal basis vectors to another 3 sets of orthonormal basis vectors[6] Note that this is not a rigorous proof but it is suggestive.

Thus, we need a higher level of abstraction to Schmidt-decompose three-qubit: we are going to make the first qubit a matrix. Let

$$|\psi\rangle = \sum_{i,j,k=0}^{n} t_{ijk} |ijk\rangle$$

[6]See Schmidt decomposition in the *Useful Theorem and Lemmas* section.

And let define the matrix

$$[T]_{ijk} \equiv t_{ijk}$$

All we mean by that is:

$$T_0 = \begin{pmatrix} t_{00}^0 & t_{01}^0 \\ t_{01}^0 & t_{11}^0 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} t_{00}^1 & t_{01}^1 \\ t_{01}^1 & t_{11}^1 \end{pmatrix}$$

where the superscript on t refers to which matrix t belongs to. Because we're working with bits and n is a fancy way to say "1."

Now, let

$$T_0' = u_{00}^A T_0 + u_{01}^A T_1$$

$$T_1' = u_{10}^A T_0 + u_{11}^A T_1$$

where the u_{ij}^A are entries of a unitary matrix U^A. We later see that the matrix corresponds to a unitary transformation on the first qubit, hence the ^{A}.

Now, make the constraint

$$\det(T_0') = 0$$

$$= \left(u_{00}t_{00}^0 + u_{01}t_{00}^1\right)\left(u_{00}t_{11}^0 + u_{01}t_{11}^1\right) - \left(u_{00}t_{10}^0 + u_{01}t_{10}^1\right)\left(u_{00}t_{01}^0 + u_{01}t_{01}^1\right) = 0$$

defining

$$x \equiv \frac{u_{01}}{u_{00}}$$

we get after dividing both sides of the equation by u_{00} and some use of determinant properties

$$x^2\left(t_{00}^1 t_{11}^1 - t_{10}^1 t_{01}^1\right) + x\left(t_{00}^1 + t_{11}^0 + t_{11}^1 t_{00}^0 - t_{01}^1 t_{10}^0 - t_{10}^1 t_{01}^0\right) + t_{00}^0 t_{11}^0 - t_{10}^0 t_{01}^0 = 0$$

which is a quadratic in x. Since we are working in $\mathbb{C}$, the quadratic must have one or two distinct solutions.

Now, assuming we have found the solution, let's proceed to the diagonalization:

Recalling:

$$T_0' = u_{00}^A T_0 + u_{01}^A T_1$$

$$T_1' = u_{10}^A T_0 + u_{11}^A T_1$$

we have the mapping induced by U^A

$$t_{ijk} \mapsto u_{i0}t^0_{jk} + u_{i1}t^1_{jk}$$

which means

$$|\psi\rangle = \sum_{i,j,k=0}^{n} t_{ijk} |i\rangle |j\rangle |k\rangle \mapsto |\psi'\rangle = \sum_{i,j,k=0}^{n} \left(\sum_{l=0}^{n} u_{il}t^l_{jk} \right) |i\rangle |j\rangle |k\rangle$$

$$|\psi'\rangle = \sum_{j,k=0}^{n} \sum_{l=0}^{n} t^l_{jk} \left(\sum_{i=0}^{n} u_{il} |i\rangle \right) |j\rangle |k\rangle$$

Remarking that we are sending $|i\rangle$ to another orthonormal basis vector,

$$= \sum_{j,k=0}^{n} \sum_{l=0}^{n} t^l_{jk} |i'\rangle |j\rangle |k\rangle$$

Since the operation we did is just a change of basis on the first qubit, we know we can freely reverse this operation. In other words, given

$$|\psi\rangle = \sum_{i,j,k=0}^{n} t_{ijk} |i'\rangle |j\rangle |k\rangle$$

$$= \sum_{i,j,k=0}^{n} \sum_{l=0}^{n} u_{il}t^l_{jk} |i\rangle |j\rangle |k\rangle$$

Now, using singular value decomposition, let

$$T'_0 = U^B M'_0 U^C$$

$$T'_1 = U^B M'_1 U^C$$

such that M'_0 is a diagonal matrix, and U^B, U^C are unitary transformations. Then, letting

$$\sum_{l=0}^{n} u_{il}t^l_{jk} = a^i_{jk}$$

we have

$$|\psi\rangle = \sum_{i,j,k=0}^{n} a^i_{jk} |i\rangle |j\rangle |k\rangle$$

$$= \sum_{i=0}^{n} \sum_{j,k=0}^{n} \sum_{r,s=0}^{n} w_{jr}^{i} m_{rs}^{i} v_{sk}^{i} |i\rangle |j\rangle |k\rangle$$

$$= \sum_{i=0}^{n} \sum_{r,s=0}^{n} \left(\sum_{j=0}^{n} w_{jr}^{i} |j\rangle \right) \left(\sum_{k=0}^{n} v_{sk}^{i} |k\rangle \right) m_{rs}^{i} |i\rangle$$

$$= \sum_{i=0}^{n} \sum_{r,s=0} m_{rs}^{i} |i\rangle \left|j'\right\rangle \left|k'\right\rangle$$

where

$$w_{jr}^{0} = w_{jr}^{1} \qquad v_{sk}^{0} = v_{sk}^{1}$$

and

$$m_{rs}^{0} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & 0 \end{pmatrix} \qquad m_{rs}^{1} = \begin{pmatrix} \lambda_0 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}$$

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