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UAM | Escuela de Doctorado

A toolbox for left-orders of low complexity

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Defense talk outline

Parts:

I. Definitions: left-orderable groups and formal languages
II. Results overview
III. Selected proof-sketches
IV. Appendix (mostly other proof-sketches)

Intended audience: Thesis committee.

(Familiarity with geometric group theory concepts, left-orderable groups, formal languages and automata theory is assumed; background will be reviewed quickly.)

What this talk is based on

Previous publications:

- *Formal language convexity in left-orderable groups.* Int. J. Algebra Comput. 30(07): 1437-1456 (2020). arXiv:1905.13001 (Su, 2020)
- *Regular left-orders on groups.* J. Comb. Algebra 6, no 3/4, pp. 265-314. arXiv:2104.04475 (Antolín, Rivas, and Su, 2022)

Note this second paper is joint work.

My PhD thesis, including some new results, will be indicated by *.

Focus: I will overview all the results, but will go deeper into my favorite results.

PART I: Background

Left-orderable groups

Definition

A group G is *left-orderable* if there exists a strict total order $\prec$ on the elements of G which is invariant under left-multiplication,

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

Ex: $(\mathbb{Z}, +)$ has a natural left-order $\prec$ given by

$$\cdots \prec -1 \prec 0 \prec 1 \prec \cdots .$$

Non-ex: Any group with an element of finite order. If $g \neq 1_G \in G$ such that $g^n = 1$, for some $n \neq 0 \in \mathbb{Z}$ and wlog $1 \prec g$, then

$$1 \prec g \prec g^2 \prec \cdots \prec g^n = 1$$

is a contradiction.

Running example: $\mathbb{Z}^2$ and the Klein bottle group

- $(\mathbb{Z}^2, +)$ lexicographic left-order $\prec$ as $\mathbb{Z} \times \mathbb{Z}$

$$(x, y) \prec (x', y') \iff x < x' \text{ or } x = x', y < y'.$$

- $K_2 = \langle a, b \mid bab = a \rangle$, $\quad ba \to ab^{-1}$ has lexicographic order $\prec'$

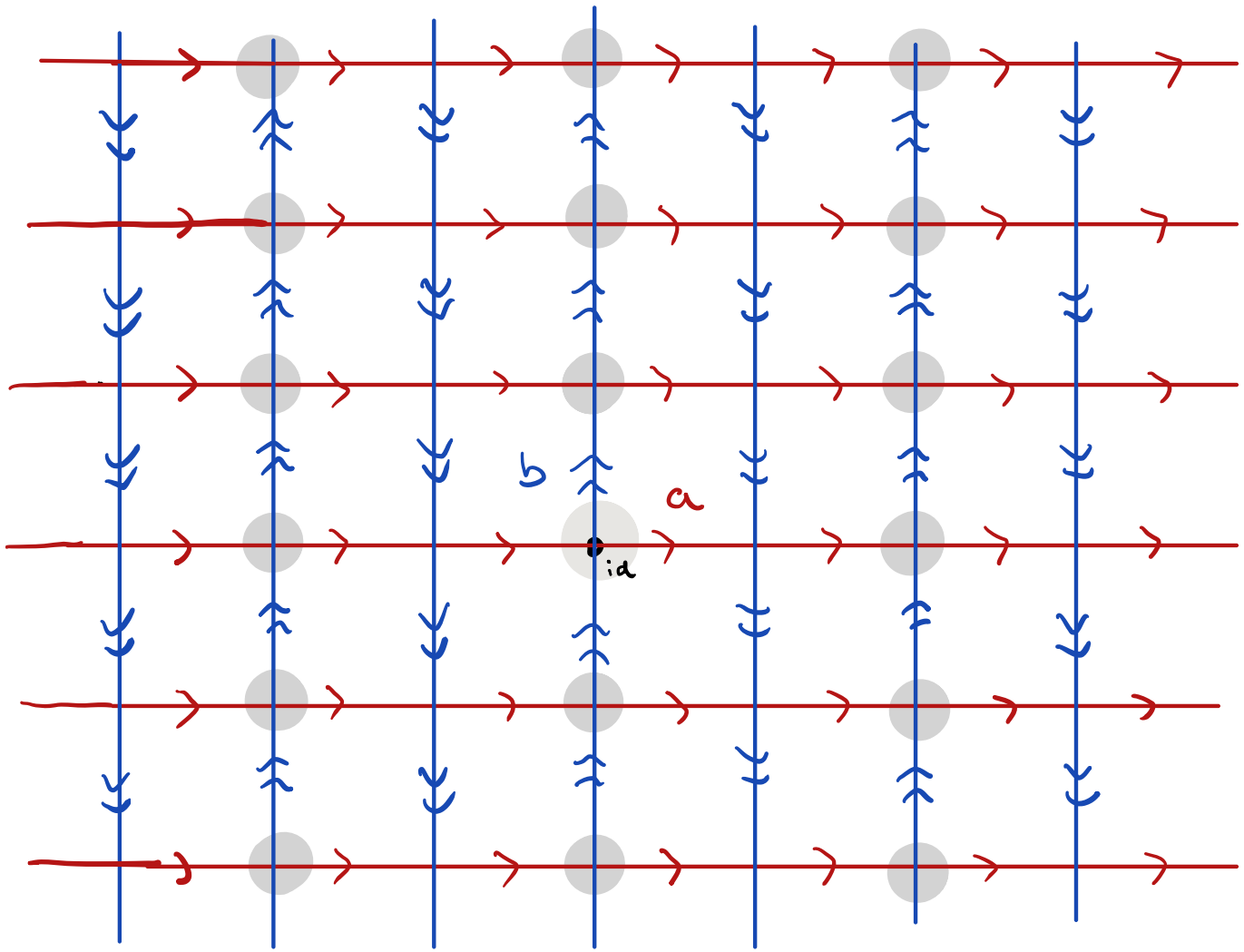
$$a^m b^n \prec' a^p b^q \iff m < p \text{ or } (m = p, n < q).$$

- Can embed $(\mathbb{Z}^2, +) \hookrightarrow K_2$ by sending $(x, y) \mapsto a^{2x} b^y$ since

$$ba^2 = (ba)a = a(b^{-1}a) = a(ab) = a^2 b.$$

Then, $\prec$ and $\prec'$ coincide on $\mathbb{Z}^2$.

Running example: $\mathbb{Z}^2 \hookrightarrow K_2$ (pictured)



Cayley graph of K_2 with $\mathbb{Z}^2$ highlighted in grey. Elements are compared lexicographically.

More examples of left-orderable groups

- $\mathbb{Z}$, $\mathbb{Z}^n$ and poly-$\mathbb{Z}$ groups,
- the lamplighter group $\mathbb{Z} \wr \mathbb{Z}$,
- free groups,
- the Klein bottle group and other surface groups,
- braid groups,
- torsion-free one-relator groups (in particular $\mathrm{BS}(1, q)$),
- RAAGs,
- subgroups of $\mathrm{Homeo}^+(\mathbb{R})$,
- locally indicable groups.

Motivations for studying left-orderable groups

- Many groups of interest in GGT are left-orderable.
- For countable groups, a left-order is equivalent to a faithful action on $\mathbb{R}$ by orientation-preserving homeomorphisms $\implies$ allows use of dynamical tools to study LO groups.
- **Hanna-Neumann conjecture (proven):** Let H and K be non-trivial free groups. Then

$$\mathrm{rk}(H \cap K) - 1 \leq (\mathrm{rk}(H) - 1)(\mathrm{rk}(K) - 1).$$

  A proof by Mieneyev, 2012 was given using left-orderable groups.
- **Zero divisor conjecture (not proven):** Let R be a ring without zero divisors and G be torsion-free. The conjecture says that the group ring RG has no zero divisors.
  - What is known is that if G is left-orderable, then RG has no zero divisors, so conjecture holds for LO groups.
  - Fisher and Sánchez-Peralta, 2024 proved the conjecture for f.g. torsion-free 3-manifold groups, Gardam, 2021 disproved the unit conjecture which is implies the zero-divisor conjecture.

Left-orderability as a decision problem

General procedure to determine whether $g \prec h$

1. Find a normal form to write each group element in.
2. Use that normal form to algorithmically decide the direction of inequality.

Ex: $(\mathbb{Z}^2 \leq K_2)$: Can determine a left-order $\prec$ by writing each elements as

$$g = a^m b^n, \quad h = a^p b^q$$

for some $m, n, p, q \in \mathbb{Z}$ and comparing them lexicographically.

Point: Comparing whether $2 <^? 3$ in $(\mathbb{Z}, +)$ is nicer than whether

$$1+1+1-1 <^? -1+1+1-1+1+1+1.$$

Positive cones

To study normal forms, easier to look at positive cones.

Definition

$P \subset G$ is a *positive cone* for G if

- $PP \subseteq P$ (semigroup closure),
- $G = P \sqcup P^{-1} \sqcup \{1_G\}$ (trichotomy).

$\sim$ Capturing notion of additive positivity in groups.

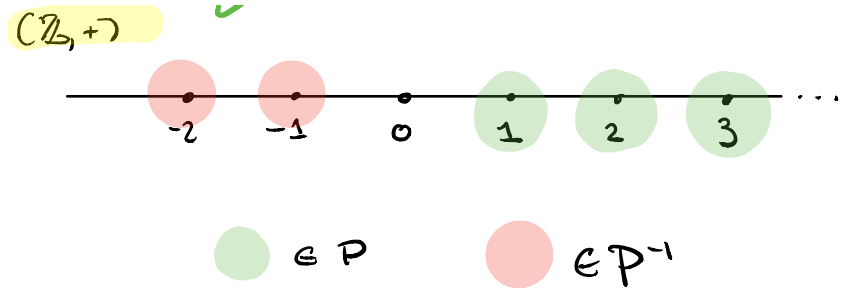
Fact: Given a left-order $\prec$, the positive cone

$$P = \{g \in G \mid g \succ 1\}$$

uniquely defines $\prec$ and vice-versa. Looking at the set of all positive elements is the same as looking at the left-order.

Positive cones 2

Ex: $(\mathbb{Z}, +)$ admits *two* positive cones, P and P^{-1}.



The left order is equivalent to defining positive elements

$$P = \{z > 0 \mid z \in \mathbb{Z}\}$$

in the sense that

$$x < y \iff -x + y > 0.$$

Positive cone languages (PCL)

Definition

A formal language $L \subseteq X^*$ which surjects to a positive cone P is called a *positive cone language* (PCL).

The normal forms of positive elements are given by the PCL.

Example PCLs for $(\mathbb{Z}, +)$:

- $L = \{x^n \mid n > 0\}$.
- $L' = \{w \in \{x, x^{-1}\}^* \mid \text{exponent sum}(w) > 0\}$.

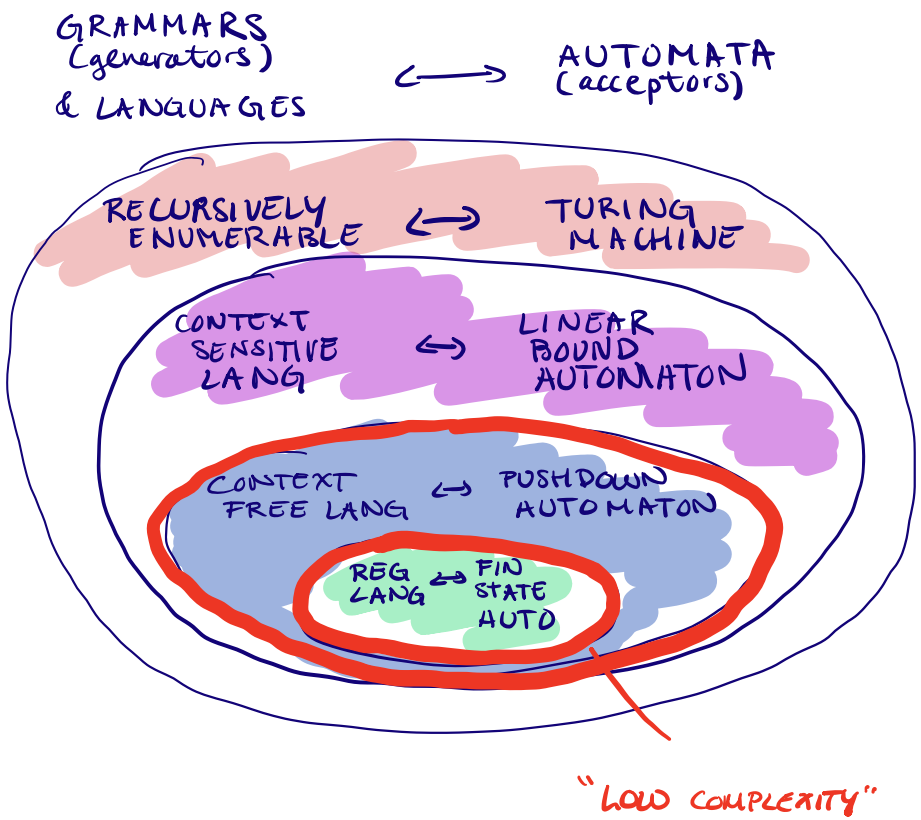
In general, we prefer L to L' because it's easier to *decide* whether a word in L represents a positive element.

- In L: need to check that we only read x's of positive power.
- In L': need to compute exponent sum and check that it's positive.

PCLs of low complexity

In this talk, we are interested in PCLs of low complexity, by which we mean PCLs for which there is simple machinery to decide whether a word belongs to the language or not.

Complexity is classified by the *Chomsky hierarchy*.



Regular languages $\leftrightarrow$ finite state automata

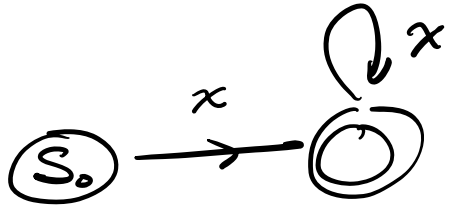
Finite State Automata (FSA)

Automata accepting language via finite memory.
Encoded by directed graphs with

- vertices ($\sim$ memory), some of which are special and labeled start states (s_0) or and accept state (concentric circle),
- edges labeled with letters of input words.

Regular language: set the words formed by the paths from start state to accept state of a FSA.

Ex: PCL for $\mathbb{Z}$, $L = \{x^n \mid n > 0\}$.



2-state memory: start (have seen nothing) and accept (have seen only x's).

Context-free languages $\leftrightarrow$ pushdown automata

Pushdown automata (PDA)

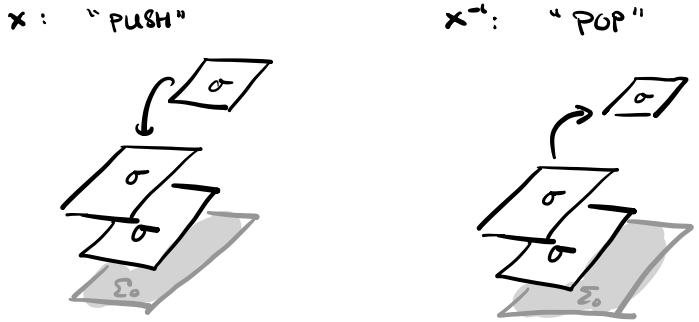
$\sim$ FSA + stack.

A *stack* (as in "stack of cards") is an infinite one-ended access tape s.t.

- can add ("push") a symbol at the top.
- can remove ("pop") a symbol at the top.

Context-free language: set of word from start to end state following the FSA and stack transitions.

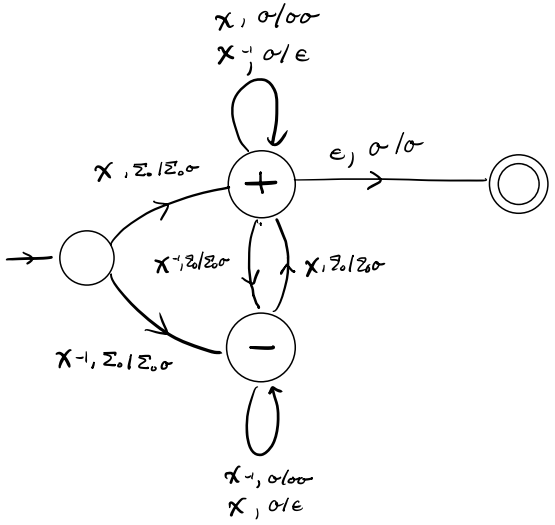
Ex ($\mathbb{Z}$): $L' = \{w \in \{x, x^{-1}\}^* \mid \text{exponent sum}(w) > 0\}$.



Push: record number of x's. Pop: match it with number of x^{-1}'s.

Context-free languages $\leftrightarrow$ pushdown automata 2:

Ex ($\mathbb{Z}$) (cont'd): $L' = \{w \in \{x, x^{-1}\}^* \mid \text{exponent sum}(w) > 0\}$.



PDA: FSA records whether sum is positive or negative (zero can be in either $+$ or -). Stack counts overflow over 0 using σ as stack symbol, and Σ_0 is the empty stack symbol denoting the bottom.

Definition

Languages accepted by PDAs with only $\{\Sigma_0, \sigma\}$ as stack symbols are called *one-counter* (1C).

Part I: End

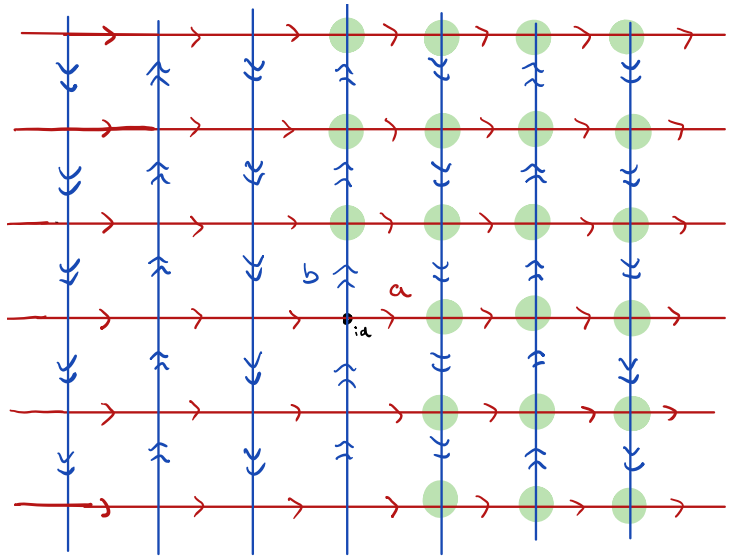
End of Part I and introductory materials.

Questions?

PART II: Results Overview

$K_2 \geq \mathbb{Z}^2$ as motivating example

Ex: $K_2 \geq \mathbb{Z}^2$ with positive cone in green.



A PCL is given by

$$L = \{a^m b^n \mid m > 0 \text{ or } (m = 0 \text{ and } n > 0)\}.$$

where if $a^m b^n \in \mathbb{Z}^2$ for m even.

Another is given by

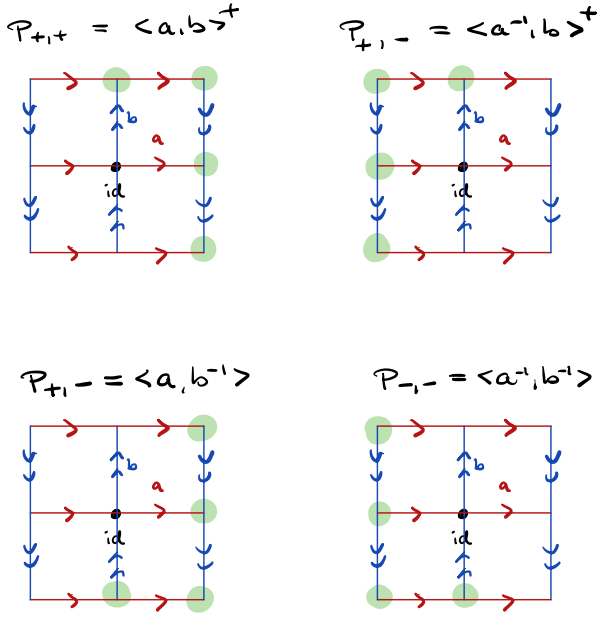
$L' = \langle a, b \rangle^+ \leftarrow$ finitely generated semigroup (i.e. grp mult. w/o inverses)!

$K_2 \geq \mathbb{Z}^2$ as motivating example 2

Fact: The positive cones of K_2 are f.g., those of $\mathbb{Z}^2$ are not.

K_2: has four f.g. positive cones given by choice of sign of generators.

$$P_{\pm,\pm} = \langle a^{\pm}, b^{\pm} \rangle^{+}$$



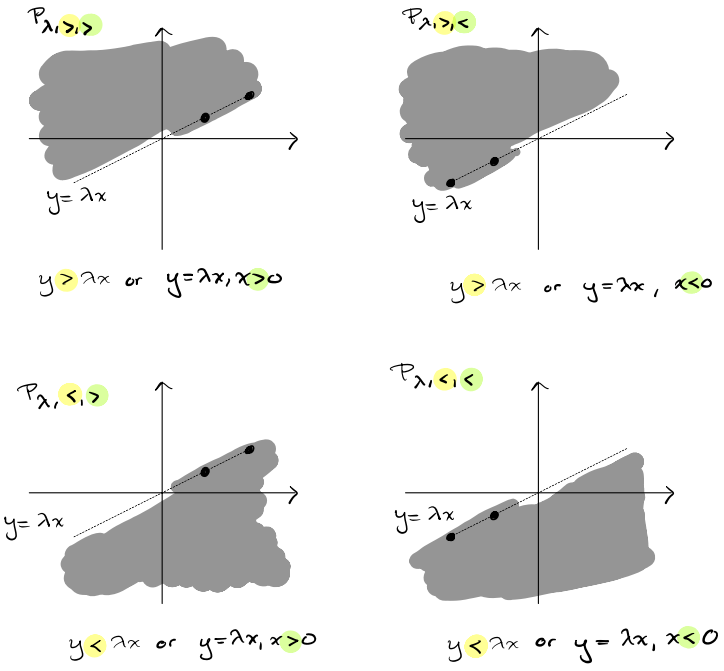
F.g. positive cones are isolated in $\mathrm{LO}(G)$ (in the sense of Sikora, 2016).

$K_2 \geq \mathbb{Z}^2$ as motivating example 3

$\mathbb{Z}^2$: has uncountably many positive cones, parametrised by a slope λ and two choices of orientation given by $\diamond_1, \diamond_2$.

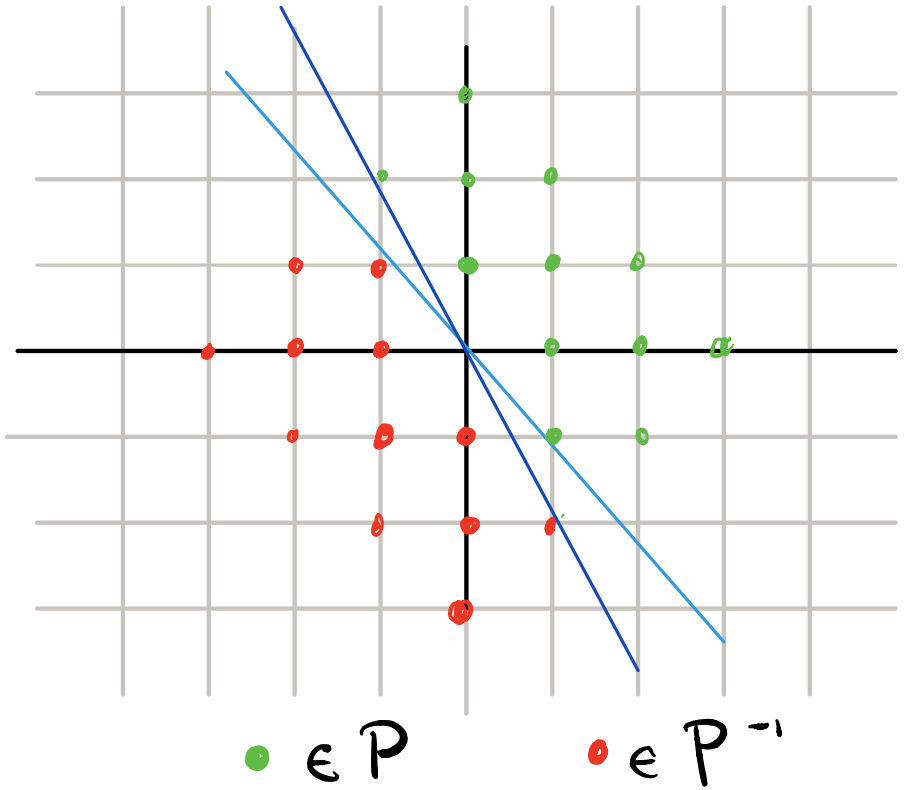
$$P_{\lambda,\diamond_1,\diamond_2} := \{(x, y) \in \mathbb{Z}^2 \mid y \diamond_1 \lambda x \text{ or } y = \lambda x, x \diamond_2 0\}$$

where $\lambda \in \mathbb{R}$, $\diamond_1, \diamond_2 \in \{<, >\}$



$K_2 \geq \mathbb{Z}^2$ as motivating example 4

$\mathbb{Z}^2$ (2): In particular, fixing $\diamond_1, \diamond_2$ and varying λ, we can get positive cones which are arbitrarily close to each other in the space of LOs.



$P_\lambda, P_\lambda{}'$ given by slopes λ, λ' in blue. Agreeing on a ball around the identity conveys a sense of closeness between two positive cones.
Bigger ball agreement $\implies$ closer. Can be done by making λ' closer to λ.

$K_2 \geq \mathbb{Z}^2$ as motivating example 5

$\mathbb{Z}^2$ (3): No positive cone is isolated because they can be made arbitrarily close as $\lambda' \to \lambda$.

K_2: has f.g. left-orders which are isolated in the space of LOs.

$\implies$ $\mathbb{Z}^2$ does not inherit f.g. positive cone as subgroup of index 2 in K_2.
$\implies$ f.g. positive cones are not inherited by finite index subgroups.

Motivation

We want a classification system for positive cone complexity which is stable under natural group-theoretic constructions.

Regular and one-counter languages allow us to do that!

Overview

Overview of results:

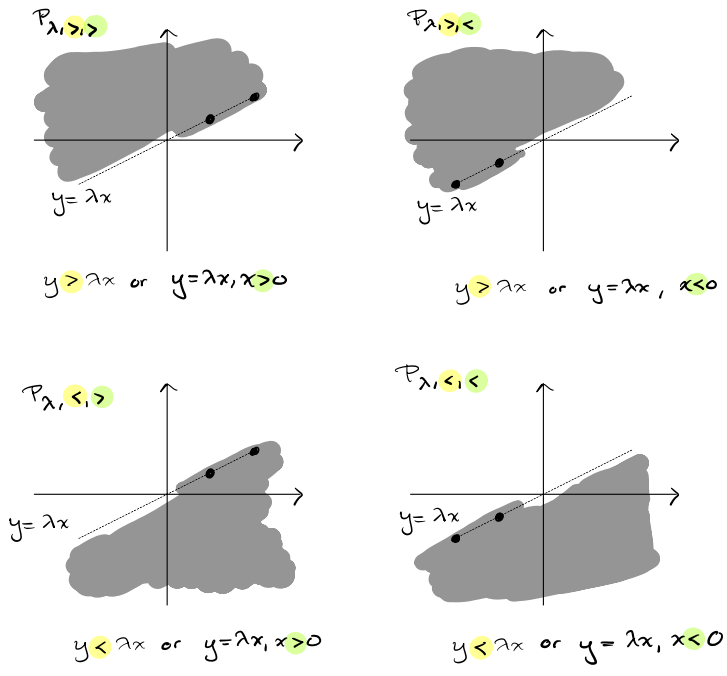
0 *Classification of positive cone complexity of $\mathbb{Z}^2$ (chap. 4)

1. PCL complexity under subgroups (chap. 10)
2. PCL complexity under extensions (chap. 11)
3. Using direct product with $\mathbb{Z}$ as a stack (chap. 13)
4. Finitely generated positive cones (chap. 12) + proof-sketches

Results 1 & 2: PCL complexity under group theoretic constructions.

Results 3 & 4: Low-complexity PCL constructions.

Misc result: Classification of LOs of $\mathbb{Z}^2$



$\lambda \in \mathbb{Q}$: reg. pos cones, all automorphic via $\mathrm{GL}_2(\mathbb{Z})$.

$\lambda \in \mathbb{R} - \mathbb{Q}$: not even context-free using an application of Parikh's lemma.

Results 1: Closure under subgroups

What subgroups inherit the PCL complexity of their overgroup?

Theorem

Language-convex subgroups inherit regular positive cones.

Definition

A subgroup $H \subseteq G$ is *language-convex* for a language L if there exists a constant R such that if for every word w in L ending in H, every point in the induced path of w in the Cayley graph of G lies within R of H.

$\sim$ quasi-convexity for words:

- Convexity: if C is convex then if path p has endpoints in C then $p \subseteq C$.
- Quasiconvexity: if p has endpoints in C then whole p is in $B_R(C)$.
- Language-convex: if w evaluates to C, then p_w in the Cayley graph is in $B_R(C)$. (Plug $H = C$ for a subgroup).

Results 1: L-convexity applications

Thm: Language-convex subgroups inherit regular positive cones.

Ex of language convex subgroups:

1. Finite index subgroups (R proportional to finite index).
   $\implies$ inherit reg. positive cones from overgroups (ex: $\mathbb{Z}^2 \leq_2 K_2$)

2. H in $G = H \rtimes \mathbb{Z}$ where G f.g. and has LO led by $\mathbb{Z}$ (pumping lem)
   $\implies$ inherit reg. positive cones from overgroups.

3. Hyperbolically embedded subgroups of torsion-free acyl. hyp. groups, such as F_2. (Morse property). $\implies$ No quasi-geodesic PCL of a f.g. acylindrically hyperbolic group is regular.
   - This generalises a result of Calegari, 2003 and partially of result of Hermiller and Šunić, 2017.
   - Similar to result of Alonso, Antolín, Brum, and Rivas, 2022.

Results 2: Closure under extensions

Theorems

1. Regular positive cones pass to extensions.
2. Regular positive cones pass to wreath products.
3. LO virtually polycyclic groups have regular PCLs.
4. The class of solvable Baumslag-Solitar groups

$$\mathrm{BS}(1,q) = \langle a, b \mid aba^{-1} = b^q \rangle$$

admits 1C LOs for all $q \in \mathbb{Z}$. Additionally admits regular LOs if and only if $q \geq -1$, *countably infinite which are automorphic for $q \geq 2$.

5. Groups which admit only reg. LOs are Tararin poly-$\mathbb{Z}$ (ex: K_2).

(4) Darbinyan, 2020 constructed a solvable group with no computable LO. We construct solvable group with easily computable LOs.

Results 3: Attaching a stack via direct product with $\mathbb{Z}$

Setup: Let $G = A \underset{C}{*} B$. Assume $C = \{1_G\}$ for simplicity.

Definition

Ordering quasi-morphism $\sim \tau$ maps $G \to \mathbb{Z}$ homomorphically up to error.

1. $\tau(g) = 0 \iff g = 1_G$,
2. $\tau(g) = -\tau(g^{-1})$,
3. $\tau(g) + \tau(h) + \tau((gh)^{-1}) \leq 1$.

Thm (Šunić, 2013): if τ is an ord. q.m. then $P = \{g \in G \mid \tau(g) > 0\}$ is a positive cone for G.

$\implies$ We have a 1C framework for deciding positivity!

Result 3: Attaching a stack via direct product with $\mathbb{Z}$ 2

Let $G = A *_C B$ be an amalgamated free product of groups admitting regular positive cones. Then,

Theorem

G admits a one-counter positive cone through ordering quasi-morphisms.

- When the amalgamation is trivial ($C = 1$), this result is optimal (Hermiller and Šunić, 2017).
- Certain free products with amalgamation also do not admit regular positive cones (Alonso, Antolín, Brum, and Rivas, 2022, Theorem 1.6).

Theorem

$G \times \mathbb{Z}$ admits a regular positive cone.

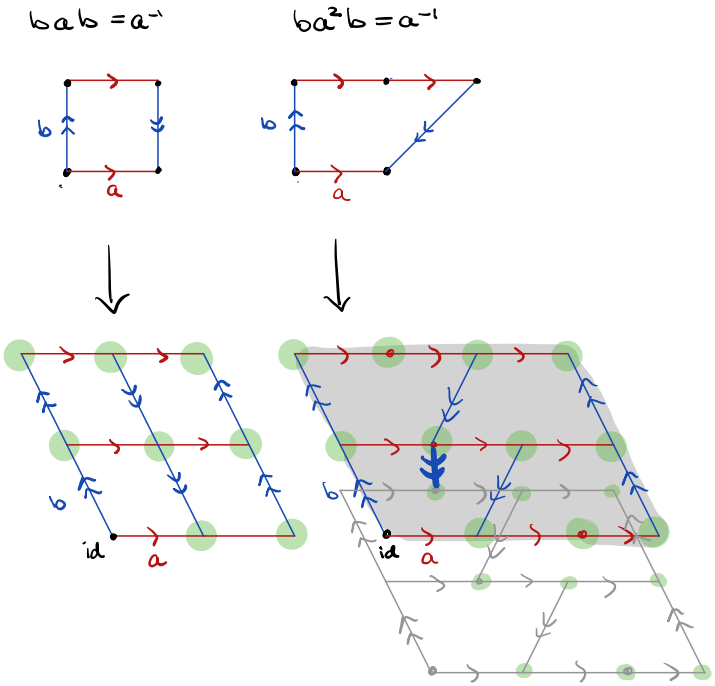
- Ex: free groups,
- Ex: Artin group whose defining graph is a tree (Hermiller and Meier, 1999),
- Ex: right-angled Artin groups based on a connected graph with no induced subgraph isomorphic to C_4 (the cycle with 4 edges) or L_3 (the line with 3 edges) (Droms, 1982).

Results 4: Groups with finitely generated positive cones

Reg and 1C positive cones are nice, but let's go back to f.g. pos. cones...

Navas, 2011 constructed an infinite family of groups containing 2-generated positive cones given by

$$n \in \mathbb{N}, \quad \Gamma_n = \langle a, b \mid ba^n b = a \rangle, \qquad P_n = \langle a, b \rangle^+.$$



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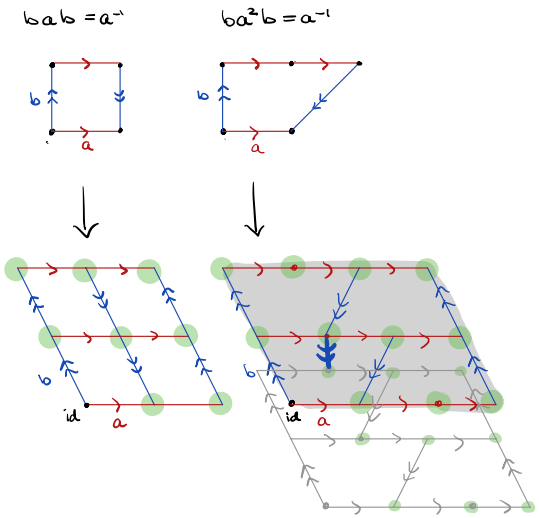


Notes:

- $\Gamma_1 = K_2$, the Klein bottle group,
- a^{n+1} commutes with everything,
- $\forall p, q \in \mathbb{Z}$, $b^p a b^q \in P_n$, i.e. pos. a-dir. $\implies$ positive.

Results 4: Groups with finitely generated positive cones 2

Navas' Q: Find an infinite family of groups with k-generated positive cones for $k \geq 3$.

Theorem

For every $k \geq 3$, we construct an infinite family of groups with k-generated positive cones.

Theorem

Let F_n be a free group of rank $n \geq 2$. Then $F_n \times \mathbb{Z}$ has a finitely generated positive cone if and only if n is even. (*new for $n > 2$).

- Hermiller and Šunić, 2017 showed that F_2 cannot have a reg. pos. cone.
- Mann and Rivas, 2018 showed that $F_2 \times \mathbb{Z}$ has an isolated order, and it was extended by Malicet, Mann, Rivas, and Triestino, 2019 to show that $F_n \times \mathbb{Z}$ has an isolated order $\iff$ n is even.
- We saw previously that f.g. pos cone $\implies$ isolated order. We show that indeed some of these orders are f.g.

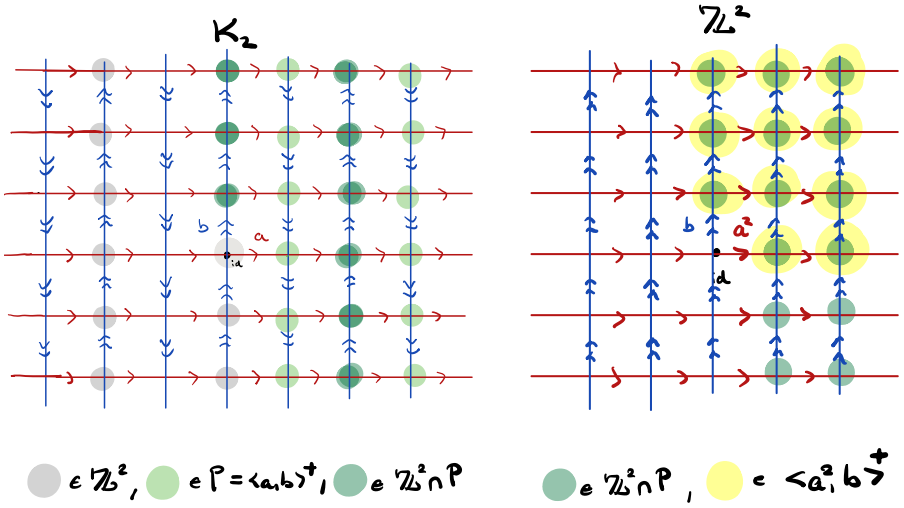
PART III: Selected proof-sketches.

Results 4: Expanding Navas' family to k-gen pos. cones

Theorem

For every $k \geq 3$, there exists an infinite family of groups with k-generated positive cones.

Proof-sketch: For $\mathbb{Z}^2 \leq K_2$, pos. cone of $\mathbb{Z}^2$ could not be f.g. because losing the b-axis twist means $\langle a^2, b \rangle^+$ only covers quarterplane.



Idea: extract f.i. subgroups by extracting Γ_n along the vertical b-direction to keep the b-twist which allows for f.g. pos. cone.

Results 4: Expanding Navas' family to k-gen pos. cones 2

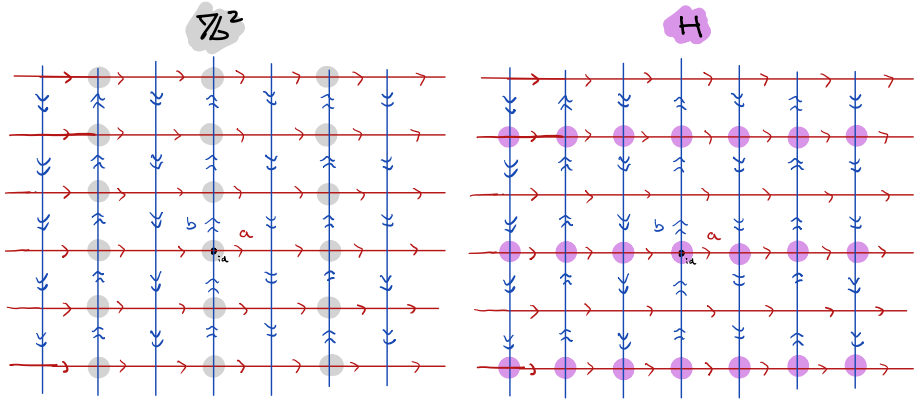
Thm: For every $k \geq 3$, we construct an infinite family of groups with k-generated positive cones.

Proof-sketch (2): To extract subgroups in different directions in Γ_1, notice that both $\mathbb{Z}^2$ and the horizontal subgroup H can be realised as kernels of maps of the form $\varphi : \Gamma_1 \to \mathbb{Z}/2\mathbb{Z}$, where

$$\varphi(baba^{-1}) \equiv 2\varphi(b) + 0\varphi(a) \equiv 0 \mod 2.$$

$\mathbb{Z}^2 = \ker \varphi, \quad \varphi(a) = 1, \quad \varphi(b) = 0, \quad$ extracts every other a,

$H = \ker \varphi \quad \varphi(a) = 0, \quad \varphi(b) = 1, \quad$ extracts every other b.



Results 4: Expanding Navas' family to k-gen pos. cones 3

Thm: For every $k \geq 3$, we construct an infinite family of groups with k-generated positive cones.

Proof-sketch (3): Previously, we obtained a subgroup with 2-gen pos cone by taking kernel of Γ_1. Try to obtain k-gen by generalizing method to $\varphi_{n,m} : \Gamma_n \to \mathbb{Z}/m\mathbb{Z}$.:

$$\varphi_{n,m}(ba^n ba^{-1}) \equiv 2\varphi(b) + (n-1)\varphi(a) \equiv 0 \mod m, \quad \varphi(b) = 1,$$

$$\iff 2 + (n-1)\varphi(a) \equiv 0 \mod m.$$

if $\gcd(n-1, m) = 1$, then $\varphi(a) = -2(n-1)^{-1} \mod m$.

Let $H_{n,m} := \ker \varphi_{n,m}$. Hard to compute $\mathrm{rk}(P_{H_{n,m}})$ directly, but since $P_{H_{n,m}}$ is a generating set for $H_{n,m}$,

$$\mathrm{rk}(P_{H_{n,m}}) \geq \mathrm{rk}(H_{n,m}) \geq \mathrm{rk}(\mathrm{Ab}\, H_{n,m})$$

and $\mathrm{rk}(\mathrm{Ab}\, H_{n,m})$ is easy to compute, can compute $H_{n,m}$ using GAP then look at its abelian rank. Iterate varying n, m.

Results 3: Expanding Navas' family: GAP simulation I

```
1   # Input: integers n,m. Output: if n != 1 and m > 1 and gcd(n-1,m) = 1 then returns kernel of phi , else return false.
2   ker_phi := function(n,m)
3   local F, a, b, Gamma_n, x, C, is_relatively_prime, mu, phi, ker_phi;
4
5   # Construct gamma_n with generators a and b.
6   F := FreeGroup("a", "b");
7   a := F.1;; b := F.2;;
8   Gamma_n := F/[b*a^n*b*a^-1];
9   a := Gamma_n.1;; b := Gamma_n.2;;
10
11  # Construct cyclic group c of order m with generator x.
12  F := FreeGroup("x");
13  x := F.1;;
14  C := F/[x^m];
15  x := C.1;;
16
17  # If n != 1 and m > 1 and gcd(n-1,m) = 1 then returns kernel of phi , else return false.
18  is_relatively_prime := GcdInt(n-1,m) = 1;
19  if is_relatively_prime and (not (n = 1)) and (m > 1) then
20          mu := -2/(n-1) mod m;
21          phi := GroupHomomorphismByImages(Gamma_n,C,[a,b],[x^mu,x]);
22          ker_phi := Kernel(phi);
23          return ker_phi;
24  else
25          return false;
26  fi; end;
27
28  # Input: a group of fp type. Output: the abelianized group.
29  abelianization := function(H)
```

Results 3: Expanding Navas' family: GAP simulation II

```
30  local iso, im_H, D, ab_H;
31  iso := IsomorphismFpGroup(H); # GAP must deduce a finite presentation for G, otherwise it refuses to do the
        computation.
32  im_H := Image(iso);
33  D := DerivedSubgroup(im_H);
34  ab_H := im_H/D;
35  return ab_H; end;
36
37  # Input: abelian group. Output: rank of abelian group.
38  rank := function(ab_H)
39  local X, rk;
40  X := GeneratorsOfGroup(ab_H);
41  rk := Size(X);
42  return rk; end;
43
44  # Input [n list] x [m list]. Output: nxm matrix with abelian ranks.
45  compute_abelian_ranks := function(n_range, m_range)
46  local n,m, H, rk, data, row;
47  data := [];
48  for n in n_range do
49          row := [];
50          for m in m_range do
51                  H := ker_phi(n,m);
52                  if not (H = false) then
53                          rk := rank(abelianization(H));
54                  else
55                          rk := -1; #If it is not possible to compute the abelianisation, then return -1 as a way of
                                showing that. This is so all the outputs can be neatly stored in an integer matrix.
56                  fi;
```

Results 3: Expanding Navas' family: GAP simulation III

```
57                      Add(row, rk);
58              od;
59              Add(data, row);
60      od;
61      Display(data);
62      return data;; end;
63
64      # Input [n list] x m. Output list of list with structure description for H_{n,m} where n varies.
65      compute_abelian_kernels := function(n_range, m)
66      local data, n, row, H, ab_H, struct_ab_H;
67      data := [];
68      for n in n_range do
69              row := [n];
70              H := ker_phi(n,m);
71              if not (H = false) then
72                      ab_H := abelianization(H);
73                      struct_ab_H := StructureDescription(ab_H);
74              else
75                      struct_ab_H := "FAIL";
76              fi;
77              Add(row, struct_ab_H);
78              Print(row, "\n");
79              Add(data, row);
80      od;
81      return data; end;
```

Results 3: Expanding Navas' family: GAP simulation 2

The data, stored in a matrix M, is as follows.

$$M_{n,m} := \begin{cases} \mathrm{rk}(\mathrm{Ab}(H_{n,m})) & \gcd(n-1,m) = 1 \\ -1 & \gcd(n-1,m) \neq 1 \end{cases}, \quad 1 \leq n, m \leq 10.$$

```
gap> n_range := [1..10];; m_range := [1..10];;
gap> data := compute_abelian_ranks(n_range, m_range);;

[ [ -1, -1, -1, -1, -1, -1, -1, -1, -1, -1 ],
  [ -1,  2,  3,  2,  2,  3,  2,  2,  3,  2 ],
  [ -1, -1,  2, -1,  2, -1,  2, -1,  2, -1 ],
  [ -1,  2, -1,  3,  5, -1,  2,  2, -1,  5 ],
  [ -1, -1,  4, -1,  2, -1,  2, -1,  4, -1 ],
  [ -1,  2,  2,  2, -1,  3,  7,  2,  2, -1 ],
  [ -1, -1, -1, -1,  3, -1,  2, -1, -1, -1 ],
  [ -1,  2,  4,  3,  2,  4, -1,  3,  9,  3 ],
  [ -1, -1,  2, -1,  6, -1,  2, -1,  2, -1 ],
  [ -1,  2, -1,  3,  3, -1,  2,  2, -1,  3 ] ]
```

Results 4: Expanding Navas' family to k-gen pos. cones 4

Thm: For every $k \geq 3$, we construct an infinite family of groups with k-generated positive cones.

Proof-sketch (4):
Data has cyclic pattern in columns $\implies$ for fixed m and variable n, cyclic pattern for $\mathrm{rk}(\mathrm{Ab}\, H_{n,m}) \leq \mathrm{rk}(P_{H_{n,m}})$. Next step is to show this formally.

Reidemeister-Schreier method: Given f.p. $G = \langle X \mid R \rangle$ and a finite Schreier transversal T for H, outputs a f.p. for H given by

$$\langle \{tx(\overline{tx})^{-1}, \quad x \in X, t \in T\} \mid \{\tau(trt^{-1}), t \in T, r \in R\} \rangle.$$

$\implies$ Set $G = \Gamma_n = \langle a, b \mid ba^n b = a \rangle$ and $T = \{b^0, b^{-1}, \ldots, b^{-(m-1)}\}$ outputs

$$Y_{n,m} := \{b^{-s} a b^{s+(m-\mu)}\}_{s=0}^{\mu-1} \cup \{b^{-s} a b^{s-\mu}\}_{s=\mu}^{m-1} \cup \{b^m\}$$

as gen set for $H_{n,m}$.

Note: By abuse of notation, we identify T with its lift $\tilde{T}$ in the universal cover.

Results 4: Expanding Navas' family to k-gen pos. cones 5

Proof-sketch (5): **Thm:** For every $k \geq 3$, we construct an infinite family of groups with k-generated positive cones.

$$Y_{n,m} := \{b^{-s}ab^{s+(m-\mu)}\}_{s=0}^{\mu-1} \cup \{b^{-s}ab^{s-\mu}\}_{s=\mu}^{m-1} \cup \{b^m\}$$

is a gen set for $H_{n,m}$.

Observations:

- Since $y \in P_n$ for all $y \in Y_{n,m}$, and every element in $\langle a, b\rangle^+ \cap H_{n,m}$ can be rewritten in $\langle Y_{n,m}\rangle^+$ due to RS method, Y generates $P_n \cap H_{n,m}$.
- By abelianizing the f.p. for $H_{n,m}$, can show that $\mathrm{rk}(\mathrm{Ab}\, H_{n,m})$ is of maximal rank $|Y_{n,m}| = m + 1$ when $n = m - 1 + mt$ for odd integer t.

$\implies$ Setting $k = m - 1$ for $k \geq 3$ proves the theorem!

Results 4: $F_2 \times \mathbb{Z}$ has f.g. positive cone

Thm: Let F_n be a free group of rank $n \geq 2$. Then $F_n \times \mathbb{Z}$ has a finitely generated positive cone if and only if n is even.

Proof-sketch: For $F_2 \times \mathbb{Z}$, start by going back to our "extracting" equation, $\varphi_{n,m} : \Gamma_n \to \mathbb{Z}/m\mathbb{Z}$ s.t.

$$\varphi_{n,m}(ba^n ba^{-1}) \equiv 2\varphi(b) + (n-1)\varphi(a) \equiv 0 \mod m, \quad \varphi(b) = 1,$$

we actually get that for $n = 2, m = 6, \varphi(a) = 4$,

$$\ker \varphi_{2,6} = H_{2,6} \cong F_2 \times \mathbb{Z}$$

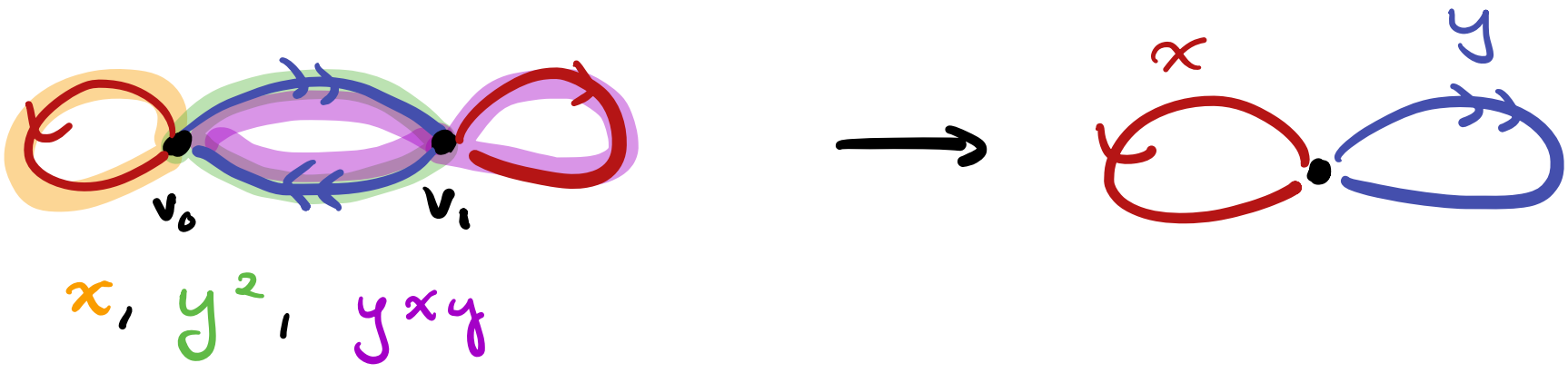
$$P_{H_{2,6}} = \langle Y_{2,6} \rangle^+, \quad Y_{2,6} := \{ab^2, b^{-1}ab^3, b^{-2}ab^4, b^{-3}ab^5, b^{-4}a, b^{-5}ab, b^6\}$$

(Note that $\mathrm{rk}(\mathrm{Ab}\, H_{2,6})$ may not be maximal here, so $\mathrm{rk}(P_{H_{2,6}}) \leq 7$.)

Results 4: $F_n \times \mathbb{Z}$ has f.g. positive cone $\iff$ n even

Thm: Let F_n be a free group of rank $n \geq 2$. Then $F_n \times \mathbb{Z}$ has a finitely generated positive cone if and only if n is even.

Proof-sketch (2): Since $F_n \leq F_2$ as a f.i. subgroup of index $n-1$ for $n \geq 2$, we can again use RS method to find a f.g. pos. cone for $F_n \times \mathbb{Z}$.



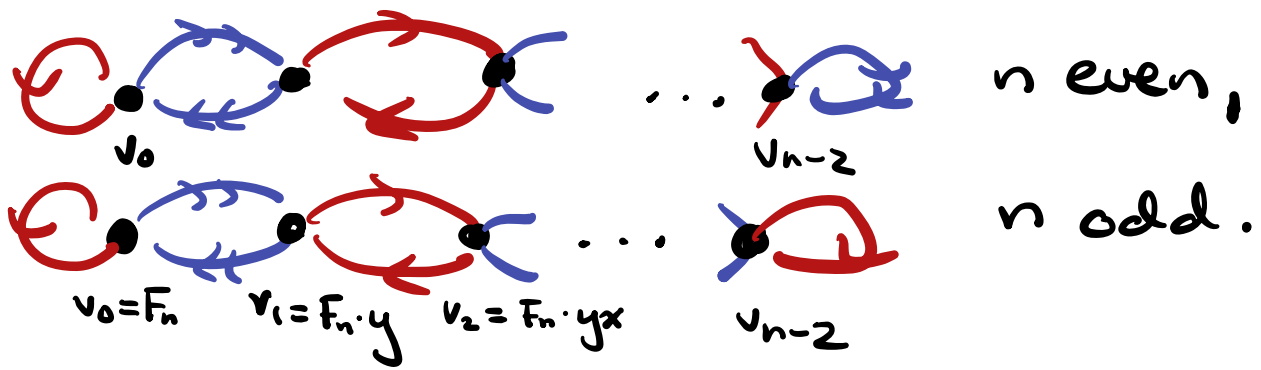
Ex: an immersion of F_3 in F_2 of index 2.

Results 4: $F_n \times \mathbb{Z}$ has f.g. positive cone $\iff$ n even 2

Thm: Let F_n be a free group of rank $n \geq 2$. Then $F_n \times \mathbb{Z}$ has a finitely generated positive cone if and only if n is even.

Proof-sketch (3):

For this choice of immersions, what's the right Schreier transversal for $F_n \leq F_2$ for the RS method?



$\Longrightarrow$ Try to find element spanning vertices of immersion.

We know from Malicet, Mann, Rivas, and Triestino, 2019 that when $n > 1$ is odd, we do not have isolated pos. cones, and thus no f.g. pos cones, so this should only work with n even.

Results 4: $F_n \times \mathbb{Z}$ has f.g. positive cone $\iff$ n even 3

Proof-sketch (4):

What's the right transversal for $F_n \leq F_2$ for the RS method?

We indeed find that element

$$g = y^{-1}x^{-1}yx$$

cyclically spans the vertices of the immersion $F_n \leq F_2 \iff n$ even.
$\implies T = \{g^0, \ldots, g^{n-2}\}$ is a Schreier transversal for $F_n \leq F_2$, n even.

Let $\psi : F_2 \times \mathbb{Z} \to H_{2,6}$ be the isomorphism map s.t.

$$x \mapsto ab^2, \quad y \mapsto a^2b^2a^2, \quad z = a^3.$$

Let $h := \psi^{-1}(b^6) = y^{-1}x^{-1}yxz^{-1}$. $\implies$ For n even,
$T = \{h^0, \ldots, h^{n-2}\}$ is a Schreier transversal for $F_n \times \mathbb{Z} \leq F_2 \times \mathbb{Z}$.

Results 4: $F_n \times \mathbb{Z}$ has f.g. positive cone $\iff$ n even 3

Thm: Let F_n be a free group of rank $n \geq 2$. Then $F_n \times \mathbb{Z}$ has a finitely generated positive cone if and only if n is even.

Proof-sketch (4): Use RS on $G = F_2 \times \mathbb{Z} = \langle Y_{2,6} \mid R' \rangle$, with $T = \{(b^6)^0, \ldots, (b^6)^{n-2}\}$.

Why it works: for n even, generating set for $F_n \times \mathbb{Z}$ is given by

$$Z = \{ty(\overline{ty})^{-1}, \quad y \in Y_{2,6}, t \in T\} \subset \{b^p a b^q\}_{p,q \in \mathbb{Z}} \cup \{1\} \subseteq P_2 \cup \{1\},$$

Then

$$|Z| \leq |Y_{2,6}| \cdot |T| \leq 7(n-1).$$

$\implies$ $F_n \times \mathbb{Z}$ has a f.g. pos. cone $\iff$ n is even.

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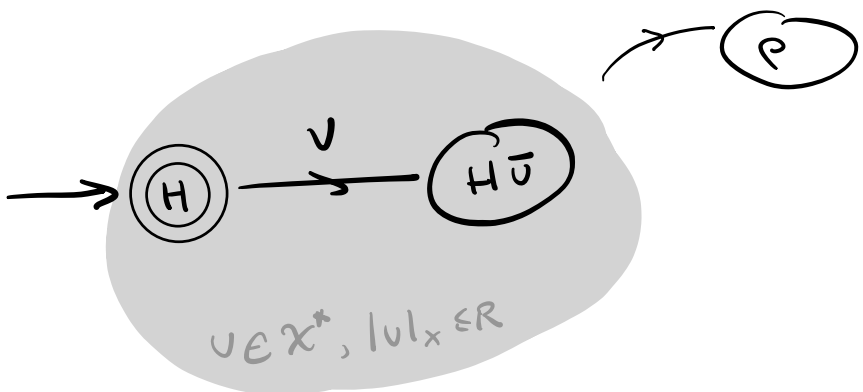
PART IV: Appendix

Results 1: L-convexity intuition

Thm: Language-convex subgroups inherit regular positive cones.

Proof-sketch: Let G have reg. PCL and H be language-convex in G.

- FSA $\sim$ finite memory. Remembering being at distance R from H only requires finite memory via finite cosets. (See figure).
- Intersecting accepted language with PCL of G picks up positive words of H by assumption of lang-conv.
- (Šunić :) Get reg. lang. in X^* evaluating to pos. cone of H. Make technical conversion to Y-gen set for H.



*Proof has been improved from original paper 1 as in fig, and subgroup PCLs are significantly easier to construct.

Results 2: Closure under extensions

Thm: Regular positive cones pass to extensions.

Proof-sketch: Let

$$1 \to N \to G \xrightarrow{f} Q \to 1.$$

be a short exact sequence (i.e. G is a Q-by-N extension). Then a positive cone for G is

$$P_G = f^{-1}(P_Q) \cup P_N.$$

Let L_Q, L_N be PCLs for P_Q, P_N respectively. Use AFL closure properties to get PCL for G.

$$L = f^{-1}(L_Q) \cup L_N$$

Results 2: Closure under wreath products

Thm: Regular positive cones pass to wreath products.

Proof-sketch: Wreath products are semi-direct products, but can't apply the ext. thm. directly since

$$N \wr Q = \oplus_{\omega \in Q} N \rtimes Q, \quad \text{s.t. } q(\oplus_{\omega \in Q} N)q^{-1} = \oplus_{q\omega \in Q} N.$$

$\implies$ write $(\mathbf{n} = \{n_q\}_{q \in Q}, p) = (\prod_{q \in Q} q n_q q^{-1})p$.

$$P_{\mathbf{N}} = \{(n_q)_{q \in Q} \in \mathbf{N} \mid n_{q'} \in P_N \text{ where } q' = \max\{q \in Q \mid n_q \neq 1_N\}\}$$

use the $\prec_Q$-order to write this element uniquely as

$$(q_1 n_1 q_1^{-1})(q_2 n_2 q_2^{-1}) \cdots (q_n n_m q_m^{-1})p, \quad q_i \succ_Q q_{i+1}, n_1 \in P_N$$

Then $L := Y^* L_N M_Q (X^* M_Q)^* Y^* \cup L_Q$ is our desired language with X, Y gen set of N and Q, L_N, L_Q, PCL for N and Q, and M_Q P^{-1}CL for Q.

Results 2: LO virtually polycyclic groups have regular LOs

Thm: LO virtually polycyclic groups have regular LOs.

Proof-sketch: These groups are poly-$\mathbb{Z}$ by induction on Hirsch length.

Hirsch length is well-defined for virtually polycyclic groups using their f.i. polycylic group. Moreover, if $N \triangleleft G$ then both N and G/N are virt. polycyclic and

$$h(G) = h(N) + h(G/N).$$

Morris theorem that all f.g. LO amenable groups (incl. LO virtually polycyclic groups) surject onto $\mathbb{Z}$.

$\implies$ $\exists$ normal subgroup N (the kernel) s.t. $G/N \cong \mathbb{Z}$

$\implies$ N is LO virt. polycyclic group with

$$h(N) = h(G) - h(G/N) = h(G) - 1.$$

$\implies$ (ind. hyp.) N is poly-$\mathbb{Z}$, and so is G.

Results 2: Classification of PCLs for BS$(1, q)$

Thm: BS$(1, q) = \langle a, b \mid aba^{-1} = b^q \rangle$ admits 1C LOs for all $q \in \mathbb{Z}$. Additionally admits regular LOs $\iff$ $q \geq -1$, *countably infinitely many which are automorphic.

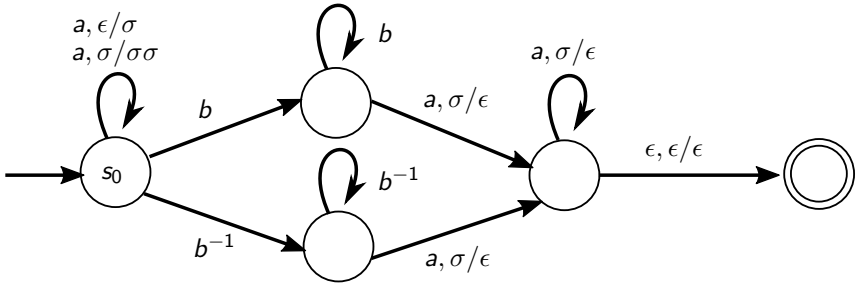
Proof-sketch: For 1C, use isomorphism

$$\text{BS}(1, q) \cong \mathbb{Z} \ltimes_\varphi \mathbb{Z}[1/q]$$

given by $\Psi(a^n \cdot a^{-t} b^r a^t) = (n, rq^{-t})$. Notice that

$$M = \{a^{-t} b^r a^t \mid t \geq 0, r \in \mathbb{Z}\}$$

is 1C (need to match t-count on LHS and RHS).



Limiting factor for complexity is M, combine it with other reg. lang. to get 1C PCLs. Lower bound the complexity (see thesis or paper).

Results 2: Classification of PCLs for BS$(1, q)$ 2

BS$(1, q)$ admits 1C LOs for all $q \in \mathbb{Z}$. Additionally admits regular LOs $\iff$ $q \geq -1$, *countably infinitely many which are automorphic.

Proof-sketch (cont'd): For a regular PCL, observe that for $q = -1, 0, 1$, BS$(1, q)$ is poly-$\mathbb{Z}$ $\implies$ we have already gone over this.

Assume $q \geq 2$, then can define a dynamical realisation on BS$(1, q)$ given by

$$\rho(a)(x) = qx, \quad \rho(b)(x) = x + 1.$$

Then,

$$\begin{aligned} Q_0 &= \{g \in \mathrm{BS}(1, q) \mid \rho(g)(0) > 0\} \cup \mathrm{Stab}^+(0) \\ &= \{a^n \mid n > 0\} \cup \{a^n(a^{-m}b^k a^m) \mid k > 0, m \geq 0, n \in \mathbb{Z}\} \end{aligned}$$

is a regular positive cone as it can be given by the PCL

$$L_0 = a^+ \cup (a|a^{-1})^* b^+ a^*.$$

Results 2: Classification of PCLs for BS(1, q) 3

Thm: BS(1, q) admits 1C LOs for all $q \in \mathbb{Z}$. Additionally admits regular LOs $\iff$ $q \geq -1$, *countably infinitely many which are automorphic.

Proof-sketch (cont'd 2):
To get countably many other regular automorphic PCLs, generalise Q_0 to

$$Q_\epsilon = \{g \in \mathrm{BS}(1,q) \mid \rho(g)(\epsilon) > \epsilon\} \cup \mathrm{Stab}^+(\epsilon),$$

for $\epsilon \in \mathbb{Z}[1/q]$ and observe that these also positive cones. They are regular because $Q_\epsilon = hQ_0h^{-1}$ where $\epsilon = \frac{r}{q^s}$ and $h = a^{-s}b^r a^s$, so

$$w_h L_0 w_{h^{-1}}$$

is our desired regular PCL.

By a result of Rivas, 2010, every LO on BS(1, q) is given by either our 1C PCL or a dynamical realisation, of which the above is regular.

Results 2: Groups which admit only regular LOs

Thm: G admits only regular LOs $\iff$ G is Tararin poly-$\mathbb{Z}$.

Definition

A *Tararin group* G is a group which admits a subnormal series

$$G = G_0 \lhd G_1 \lhd \cdots \lhd G_n = \{1\}$$

such that G_i/G_{i+1} are rank one abelian for $i = 1, \ldots, n-1$, and G_i/G_{i+2} is not bi-orderable for $i = 1, \ldots, n-2$.

If G is LO, has 2^n orders given by choice of sign of quotients G_i/G_{i+1}.

Proof-sketch ($\implies$) The number of regular languages are countable since reg lang $\leftrightarrow$ reg expr, which is a finite string and the number of finite strings is countable.

A result of Linnell, 2001 says that if a group admits infinitely many LOs, then it admits uncountably many. Conversely, if a group admits countably many LOs, it must admits finitely many $\iff$ it is Tararin (Tararin, 1991).

For poly-$\mathbb{Z}$ G f.g. and G_0/G_1 is abelian rank one, $G = G_0 = G_1 \rtimes \mathbb{Z}$, where G_1 is L-convex for L reg. by assumption, so G_1 has reg. PCL by Results 1. Use induction to show G is poly-$\mathbb{Z}$.

($\impliedby$) Poly-$\mathbb{Z}$ groups have regular PCLs by extension theorem above.

Results 4: ordering quasi-morphism 2

(**Setup:** Let $G = A *_C B$. Assume $C = \{1_G\}$ for simplicity. Then G admits ord. q.m.)

Explicit ord. q.m.: Write every element of G in normal form as irreducible product of a's and b's.

$$g = a^{m_1} b^{n_1} \dots a^{m_\ell} b^{n_\ell}$$

Then, an ord. q.m is given by

$$\begin{aligned}\tau(g) := \sharp(\text{positive syllables in } g) - \sharp(\text{negative syllables in } g) \\ + \sharp(\text{index jumps in } g) - \sharp(\text{index drops in } g).\end{aligned}$$

Pos. syl.: whether syl. is pos. in factors A or B.
Index jump: if $A < B$, then syl $A \to B$ is jump.

Results 4: ordering quasi-morphism 3

Ex computing ord. q.m: $G = F_2 = \langle a \rangle * \langle b \rangle$.

Choose $P_A = \langle a \rangle^+, P_B = \langle b \rangle^+$ and $A < B$.

Suppose $g = a^2 b^{-1} a^{-3}$. Then,

- positive syllables in g: $a^2 \implies \sharp = 1$,
- negative syllables in g: $b^{-1}, a^{-3} \implies \sharp = 2$,
- index jumps in g: $a^2 b^{-1} \implies \sharp = 1$
- index drops in g: $b^{-1} a^{-3} \implies \sharp = 1$.

$$\begin{aligned}\tau(g) &:= \sharp(\text{positive syllables in } g) - \sharp(\text{negative syllables in } g) \\ &\quad + \sharp(\text{index jumps in } g) - \sharp(\text{index drops in } g) \\ &= 1 - 2 + 1 - 1 = -1 \\ &< 0\end{aligned}$$

$\implies g \in P^{-1}$.

Results 4: G has 1C pos. cones

Thm: $G = A * B$ where A, B have reg. PCLs admits a one-counter positive cone through ordering quasi-morphisms.

Proof-sketch: We can count whether integer sum $\tau(g) >^{?} 0$ via exponent-sum 1C PDA (Part I). Combine with FSA's of P_A and P_B to get 1C τ-counter by incrementing/decrementing when changing factors FSA.

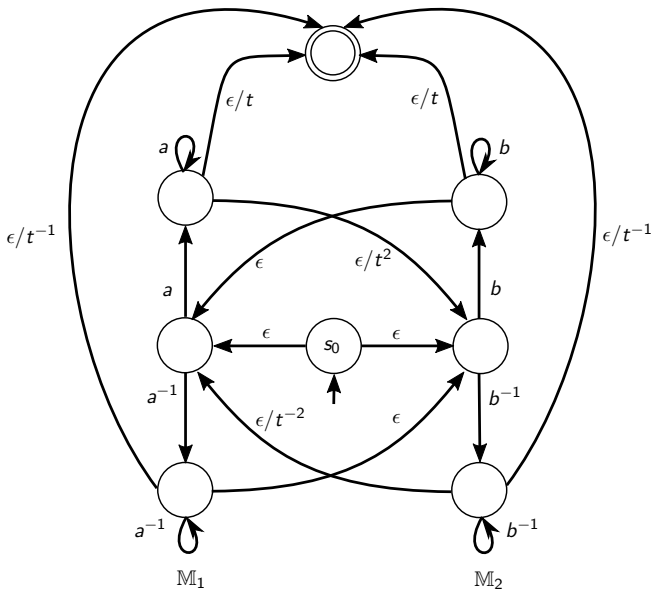


Fig: $G = F_2$, transducer outputs t, t^{-1}, this is combined to the 1C positivity counter via transducer pre-image and still 1C (see paper 2 or thesis).

Results 4: $G \times \mathbb{Z}$ has reg pos. cones

Thm: $G \times \mathbb{Z}$ admits a regular positive cone.

Proof-sketch: Had G with explicit $\tau : G \to \mathbb{Z}$ defining a positive cone when $\tau(g) > 0$. Want similar positive cone for $G \times \mathbb{Z}$.

Solution: defining

$$\tau'(g, n) = \tau(g) + 2n$$

defines an ord. q.m. for $G \times \mathbb{Z}$.

(Explicit τ is odd unless input is 1_G, and this is an important property to keep.)

Results 4: $G \times \mathbb{Z}$ has reg pos. cones 2

(**Thm:** $G \times \mathbb{Z}$ admits a regular positive cone.)

Proof-sketch (2): FSA idea for $G \times \mathbb{Z}$: append counterweight z, z^{-1}

to every arrow instead of σ^2 in the 1C τ-counter of P_G, such that every accepted word w is of the form

$$w = vz^n, \quad \tau'(v) \sim 0, \quad n \geq 0.$$

For example,

$$gz^{-\tau(g)/2} \cdot z^n, \qquad \tau'(gz^{-\tau(g)/2} \cdot z^n) = \underbrace{\tau(g) - 2\tau(g)/2}_{\sim 0} + 2n = 2n$$

where $n > 0$.

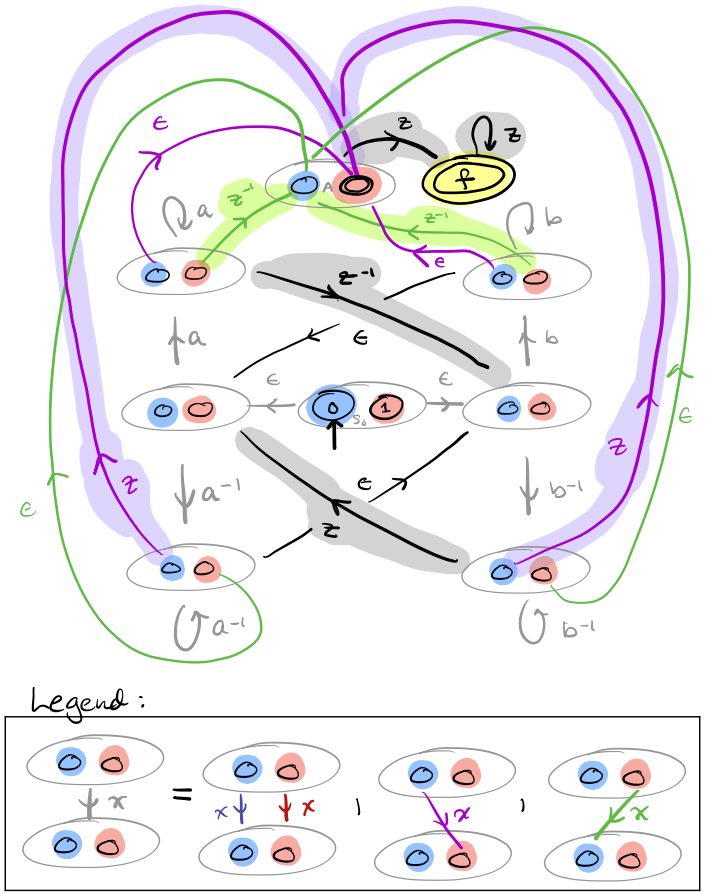
Problem: $\tau(g)$ is odd, so $\tau(g) + 2n$ is odd, not zero.

Solution: Use FSA finite memory to memorize the offset of 0 or 1.

Results 4: $G \times \mathbb{Z}$ has reg pos. cones 3

(**Thm:** $G \times \mathbb{Z}$ admits a regular positive cone.)

Proof-sketch (3): FSA for $F_2 \times \mathbb{Z}$.
Blue is 0 offset, red is 1 offset. Notice z-transitions at final state (yellow).



The end

THE END.